Stochastic analysis of (sub-pixel) geometric image processing using B-splines

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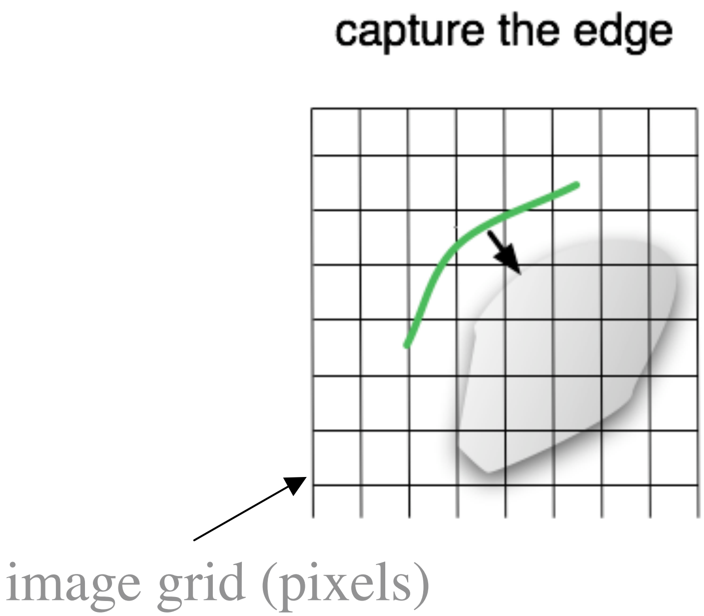
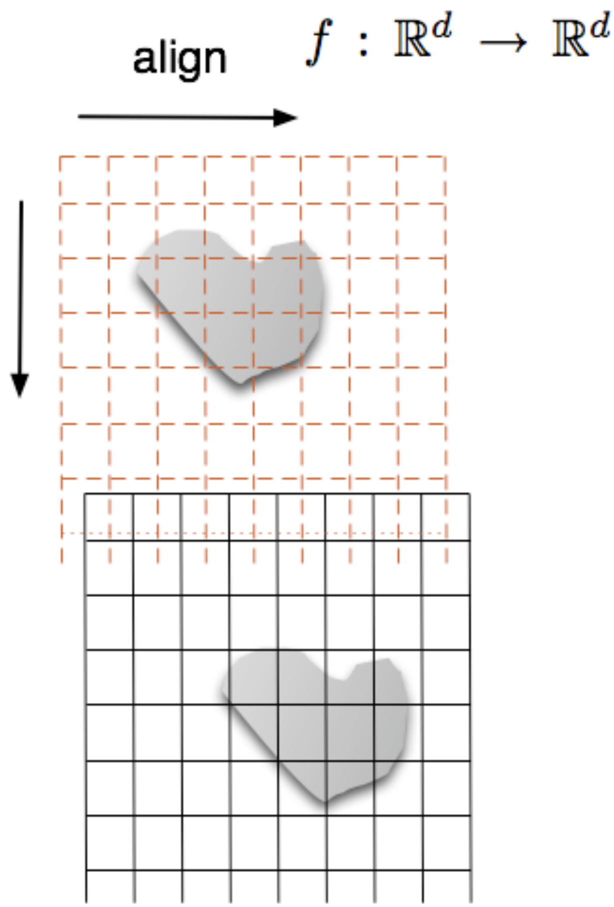
Sub-pixel geometric image processing



$$f(\gamma) = (x(\gamma), y(\gamma))$$



Variational energy minimization framework

$$\arg\min_f \Psi(s_1(\mathbf{x}), s_2(\mathbf{x}), \cdots, f) = \underbrace{\Psi_{data}(s_1(\mathbf{x}), s_2(\mathbf{x}), \cdots, f)}_{\text{Image force term}} + \Psi_{constraint}(f)$$

- registration (alignment): $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$
- edge detection: $f(\gamma) = (x(\gamma), y(\gamma))$

Image registration

$$\Psi_{data}(s_1(\mathbf{x}), s_2(\mathbf{x}), \cdots, f) = \int_{\mathbb{R}^d} \Upsilon(\mathbf{x}, f) d\mathbf{x}$$

$$\Upsilon(\mathbf{x}, f) = \frac{1}{2}\left(s_1(\mathbf{x} + \mathbf{v}(\mathbf{x})) - s_2(\mathbf{x})\right)^2$$

$$f(\mathbf{x}) = \mathbf{x} + \mathbf{v}(\mathbf{x})$$

Discretize first variation

$$\left(s_1(\mathbf{i} + \mathbf{v}(\mathbf{i})) - s_2(\mathbf{i})\right) \nabla s_1|_{\mathbf{i}+\mathbf{v}(\mathbf{i})} = 0$$

$$\Psi_{data}(s_1(\mathbf{x}), s_2(\mathbf{x}), \cdots, f) \simeq \sum_{\mathbf{i} \in \mathbb{Z}^d} \left(s_1(\mathbf{i}) - s_2(\mathbf{i} + \mathbf{v}(\mathbf{i}))\right)^2$$

Differentiate sum

- Sampled digital images:

$$\mathbf{S}_1 = s_1(\mathbf{i}),\ \mathbf{i} \in \mathbb{Z}^d$$

$$\mathbf{S}_2 = s_2(\mathbf{i}),\ \mathbf{i} \in \mathbb{Z}^d$$

- Spatial transformation:

$$\mathbf{FS} = \hat{s}(f(\mathbf{q})) = \sum_{\mathbf{i} \in \mathbb{Z}^d} s(\mathbf{i})\eta(f(\mathbf{q}) - \mathbf{i}),\ \text{with } f : \mathbb{R}^d \to \mathbb{R}^d$$

- Image registration minimization problem:

$$\min_f \|\mathbf{FS}_1 - \mathbf{S}_2\|^2$$

Edge detection

$$\Psi_{data}(s(\mathbf{x}), f) = \int_0^1 P(f, \gamma) d\gamma$$

$$P(f, \gamma) = |\nabla s(f(\gamma))|^2$$

$$f(i\tau) = \mathbf{X}_i, \; i = 0, \cdots, L-1$$

Discretize first variation

$$\nabla P|_{\mathbf{x}_i} = 0$$

$$\Psi_{data}(s(\mathbf{x}), f) = -\tau \sum_{i=0}^{L-1} |\nabla s(f(i\tau))|^2, \; \tau = \frac{1}{L}$$

Differentiate sum

- Edge (contour) function:

$$f(\gamma) = (x(\gamma), y(\gamma))$$

- Edge detection minimization problem:

$$\min_f -\tau \sum_{i=0}^{L-1} |\nabla s(f(i\tau))|^2, \ \tau = \frac{1}{L}$$

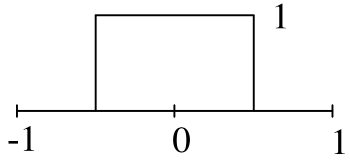
Derivatives computed based on interpolation/approximation

Image interpolation

- B-splines:

$$\beta^n = \beta^{n-1} * \beta^0$$

$\beta^0(\mathbf{x})$ →

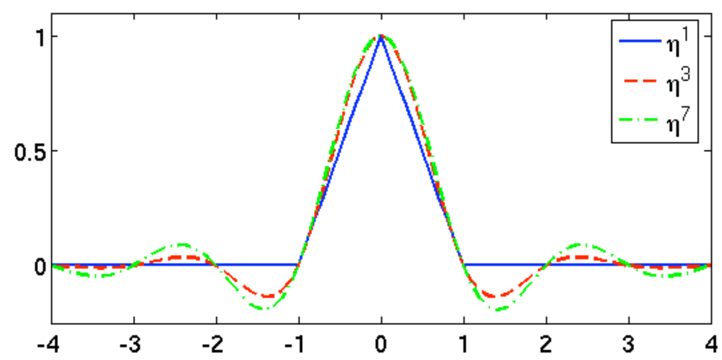


- Image interpolation:

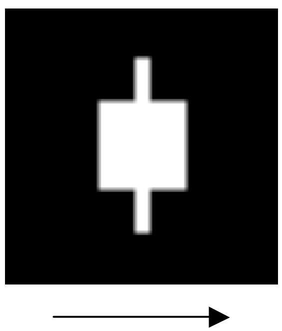
$$\hat{s}(\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{Z}^d} s(\mathbf{i}) \eta^n(\mathbf{x} - \mathbf{i})$$

- Cardinal splines:

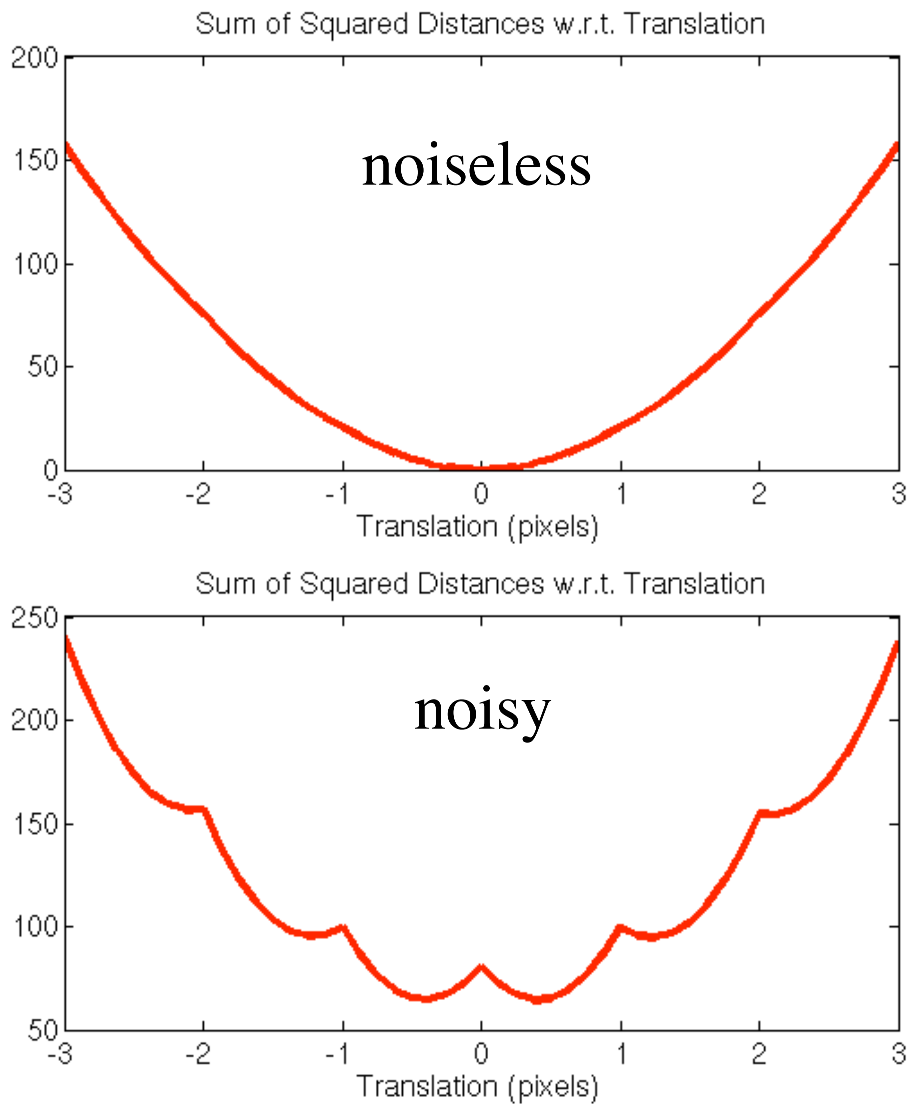
$$\eta^n(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} (\beta^n)^{-1}(\mathbf{k}) \beta^n(\mathbf{x} - \mathbf{k})$$



An interesting experiment



translate w.r.t. itself



Stochastic image model

- Additive noise: $s(\mathbf{i}) = w_s(\mathbf{i}) + e(\mathbf{i})$

- Image covariance:

$$\begin{aligned} R_s(\mathbf{i},\mathbf{j}) &= \mathrm{E}\{(s(\mathbf{i}) - \overline{s}(\mathbf{i}))(s(\mathbf{j}) - \overline{s}(\mathbf{j}))\} \\ &= \mathrm{Cov}\{e(\mathbf{i}), e(\mathbf{j})\} = R_e(\mathbf{i},\mathbf{j}) \end{aligned}$$

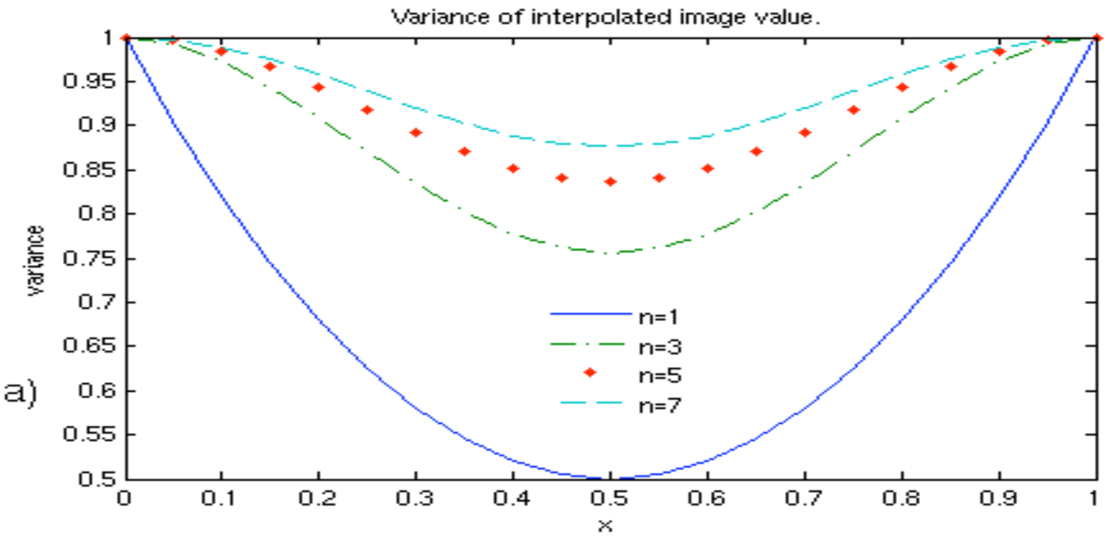
- Interpolated covariance (white noise):

$$R_{\hat{s}}(\mathbf{x},\mathbf{x}) = \sigma^2 \sum_{\mathbf{q}\in\mathbb{Z}^d} [\eta^n(\mathbf{x}-\mathbf{q})]^2$$

$$R_{\hat{s}^k}(\mathbf{x},\mathbf{x}) = \sigma^2 \sum_{\mathbf{q}\in\mathbb{Z}^d} \left[\frac{d\eta^n}{dx_k}(\mathbf{x}-\mathbf{q})\right]^2 \qquad \hat{s}^k = \frac{d\hat{s}(\mathbf{x})}{dx_k}$$

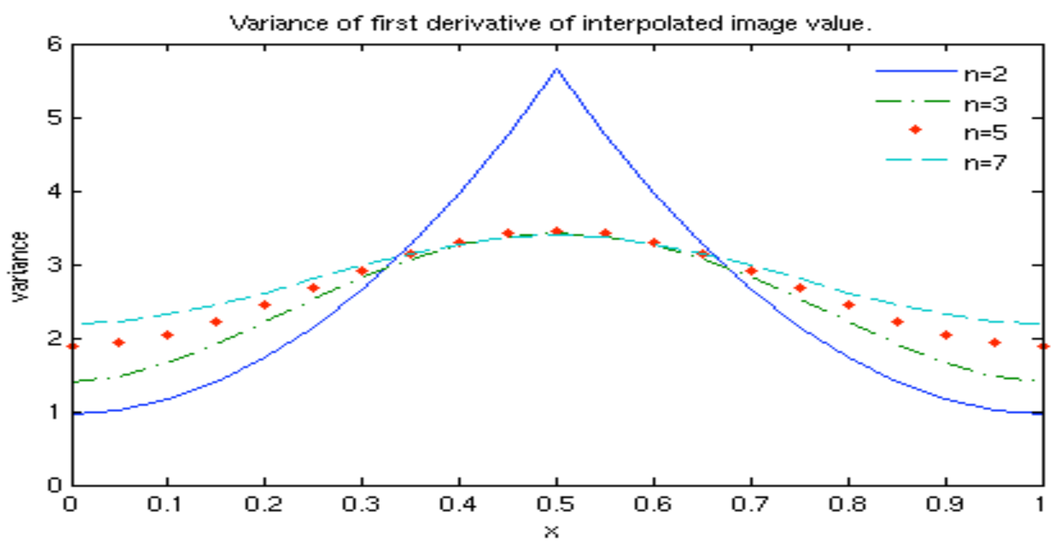
$$R_{\hat{s}}(\mathbf{x},\mathbf{x}) = \sigma^2 \sum_{\mathbf{q}\in\mathbb{Z}^d} \left[\eta^n(\mathbf{x}-\mathbf{q})\right]^2$$

a)

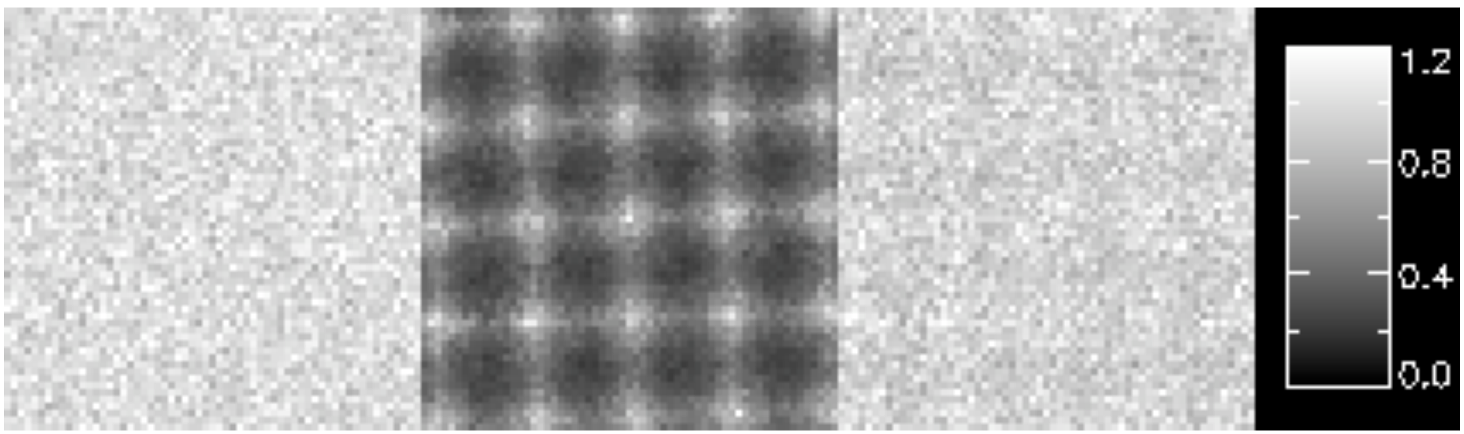


$$R_{\hat{s}^k}(\mathbf{x},\mathbf{x}) = \sigma^2 \sum_{\mathbf{q}\in\mathbb{Z}^d} \left[\frac{d\eta^n}{dx_k}(\mathbf{x}-\mathbf{q})\right]^2$$

b)



Simulation



Linear interpolation   Sinc interpolation

Deconstructing 2 norms

- Image registration:

$$\|\mathbf{F}_v\mathbf{s}-\mathbf{t}\|^2 = \|\mathbf{F}_v\tilde{\mathbf{w}}_\mathbf{s} - \tilde{\mathbf{w}}_\mathbf{t}\|^2 + \|\mathbf{e}_\mathbf{t}\|^2 + \boxed{\|\mathbf{F}_v\mathbf{e}_\mathbf{s}\|^2} + \mathcal{G}$$

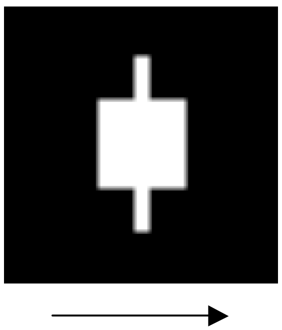
- Edge detection:

$$\sum_{i=0}^{L-1} |\nabla s(f(i\tau))|^2 = -\sum_{i=0}^{L-1} \|\nabla \hat{w}_s(f(i\tau))\|^2 - \boxed{\|\nabla \hat{e}(f(i\tau))\|^2} + \mathcal{Q}$$

$$\frac{1}{N^d}\|\mathbf{F}_v\mathbf{e}_\mathbf{s}\|^2 \approx R_{\hat{\mathfrak{s}}} \longrightarrow$$ image sample variance (uniform translation)

$$\frac{1}{L}\sum_{i=0}^{L-1} \|\nabla \hat{e}(f(i\tau))\|^2 \approx R_{\hat{\mathfrak{s}}k} \longrightarrow$$ derivative sample variance (st. line)

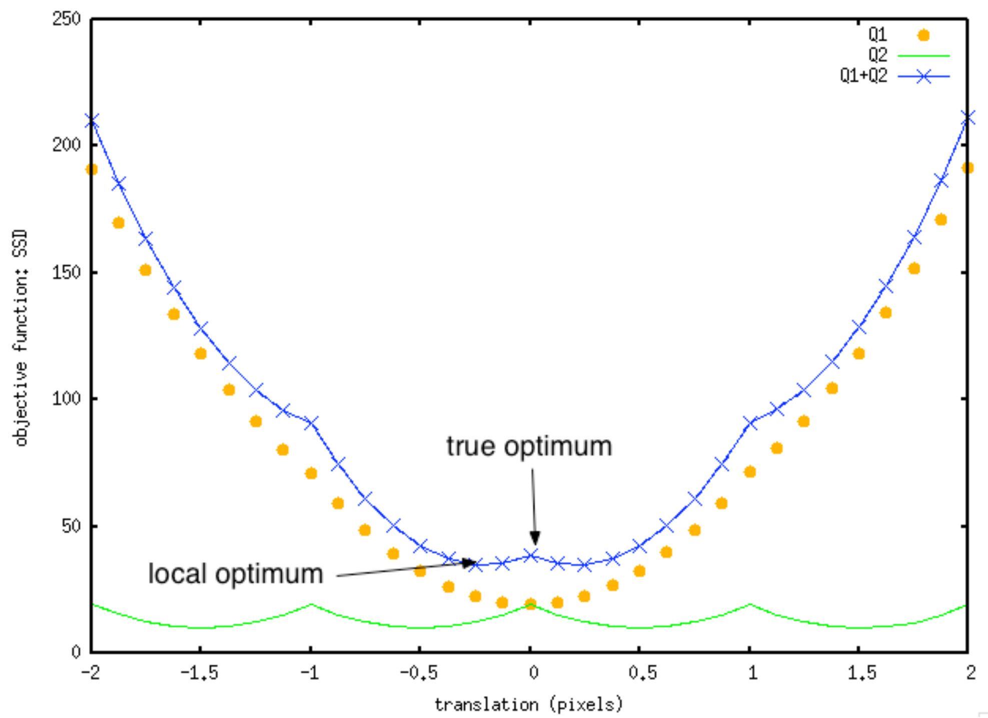
Back to intriguing experiment



$$\|\mathbf{F}_v\mathbf{s} - \mathbf{t}\|^2 = Q_1 + Q_2$$

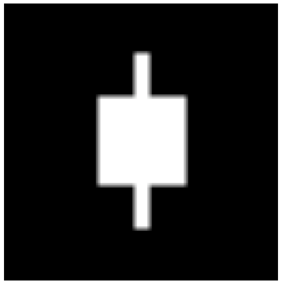
$$Q_1 = \|\mathbf{F}_v\tilde{\mathbf{w}}_\mathbf{s} - \tilde{\mathbf{w}}_\mathbf{t}\|^2 + \|\mathbf{e}_\mathbf{t}\|^2$$

$$Q_2 = \|\mathbf{F}_v\mathbf{e}_\mathbf{s}\|^2$$



Rotation too

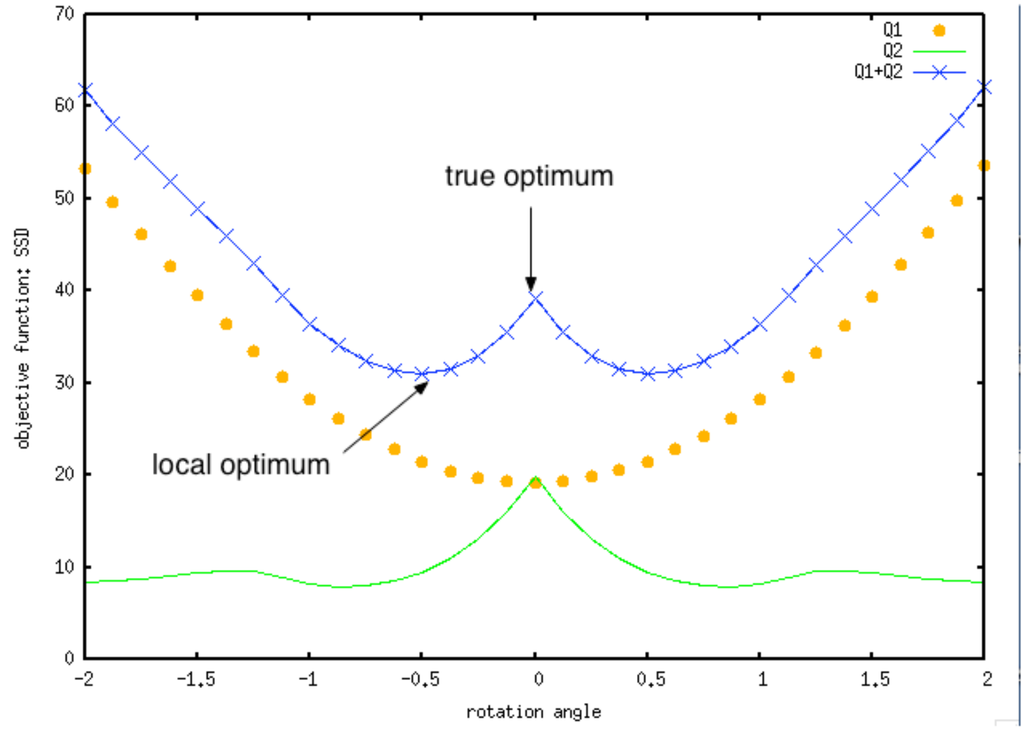
rotate



$$\|\mathbf{F}_v \mathbf{s} - \mathbf{t}\|^2 = Q_1 + Q_2$$

$$Q_1 = \|\mathbf{F}_v \tilde{\mathbf{w}}_\mathbf{s} - \tilde{\mathbf{w}}_\mathbf{t}\|^2 + \|\mathbf{e}_\mathbf{t}\|^2$$

$$Q_2 = \|\mathbf{F}_v \mathbf{e}_\mathbf{s}\|^2$$



Other similarity measures (cost functions)

- Cross correlation:

$$\Psi(\hat{s}(f(\mathbf{q})), t(\mathbf{q}), f) = \frac{(\mathbf{F}_v \mathbf{s})^T \mathbf{t}}{\|\mathbf{F}_v \mathbf{s}\| \, \|\mathbf{t}\|}$$

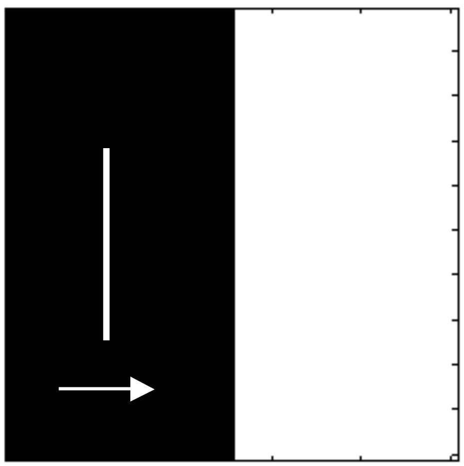
$$\sqrt{\|\mathbf{F}_v \mathbf{s}\|^2} = \sqrt{\|\mathbf{F}_v \tilde{\mathbf{w}}_\mathbf{s}\|^2 + \mathbf{F}_v \mathbf{e}_\mathbf{s}\|^2}$$

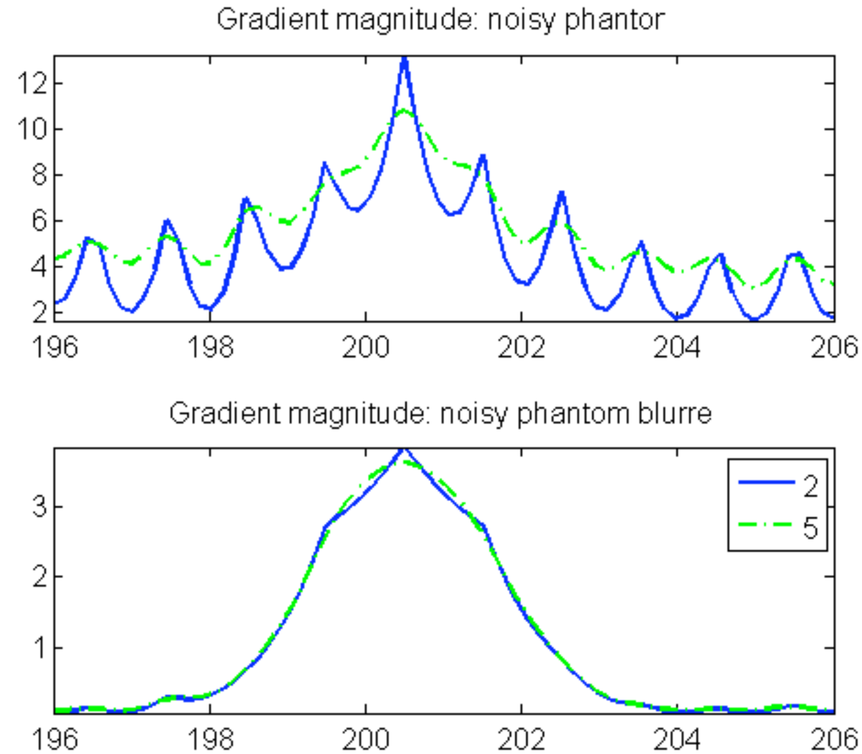
- Mutual Information (Gaussian assumption):

$$\Psi(\hat{s}(f(\mathbf{q})), t(\mathbf{q}), f) = -\frac{1}{2} \log\left(1 - \rho^2\right)$$

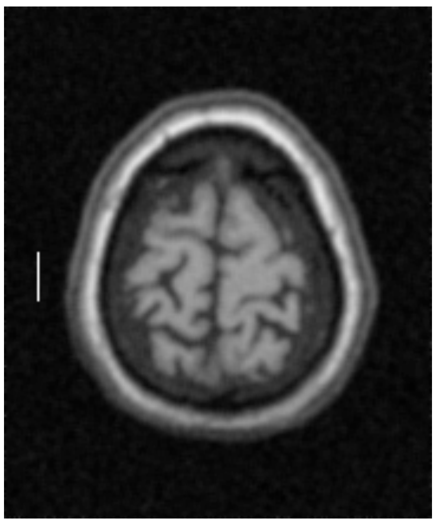
correlation coefficient

Edge detection

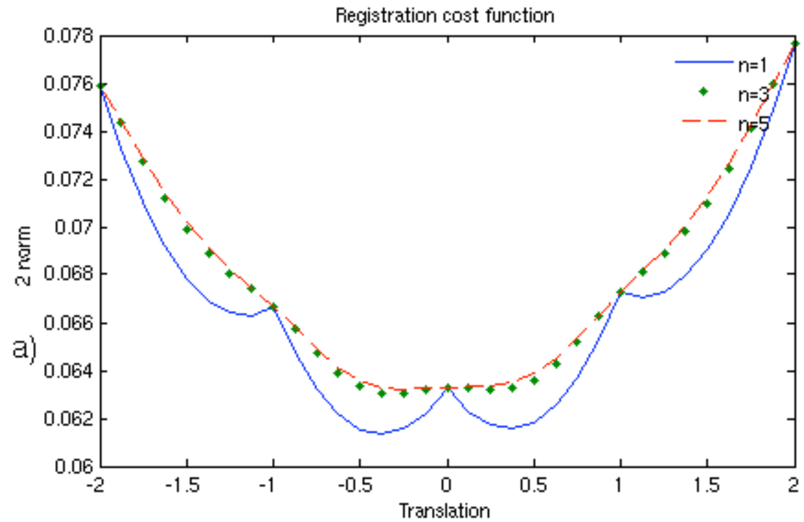




Real data:

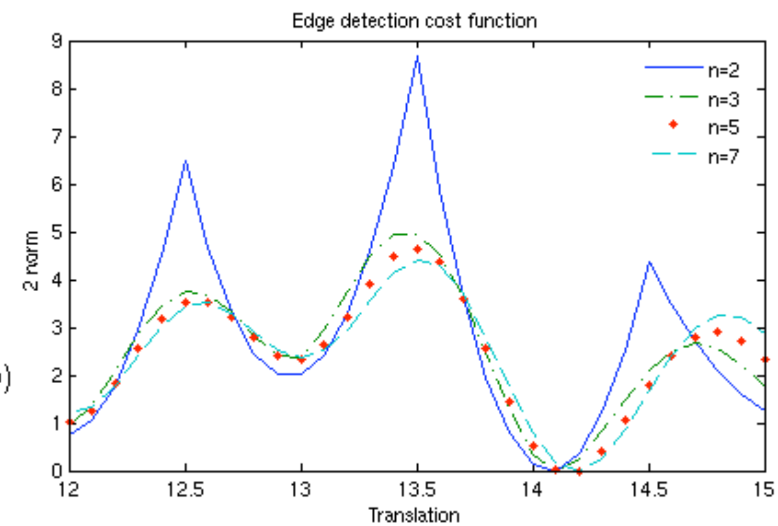


a)



Registration cost function
2 norm
Translation
n=1
n=3
n=5

b)



Edge detection cost function
2 norm
Translation
n=2
n=3
n=5
n=7

Summary & Conclusions

- Trying to obtain information about geometry to sub-pixel accuracy from discrete images is problematic: local optima artifacts.
- Provided a stochastic description for artifacts in terms of the variance/covariance of B-spline interpolated images.
- Solutions: blurring, higher degree (sinc) interpolation.