PSEUDO-RATIONALIZABILITY OVER INFINITE CHOICE SPACES

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ABSTRACT. A dominant concern in rational choice theory is *rationalizability* of choice functions. Rationalizability demands that a choice function can be represented by a preference relation according to which the choice function selects for each decision problem those alternatives which are most preferred (or undominated by other alternatives). A less demanding notion demands that a choice function can be represented by a *collection* of preference relations according to which for each decision problem the choice function selects those alternatives which are best (or undominated) for *at least one* preference relation in the given collection. This notion, called *pseudo-rationalizability*, has been given special attention in social choice theory and by philosophers such as Hans Rott and Isaac Levi.

The purpose of this article is to extend known representation results due to Aizerman and Malishevski [1] and Moulin [5]. In [1] and [5] it is assumed that the underlying set of all options X is *finite* and the domain of the choice function consists of all *finite* nonempty subsets of X. We extend these results, relaxing both assumptions.

Summary of Present Contents: In Section 1 we review choice functions and introduce some domain and functional constraints we sometimes assume in the paper. In Section 2 we review some properties of binary relations and state and prove several variants of Spzilrajn's Theorem (these will become useful in later versions of the article). Finally, in Section 3 we introduce the notion of *pseudo-rationalizablility* and state and prove several results which extend known pseudo-rationalizability results. Further extensions of the results due to Aizerman and Malishevski and to Moulin will be added to this article.

1. CHOICE FUNCTIONS

We begin with several definitions.

Definition 1.1. Let X be a nonempty set, and let $\mathscr{S}$ be a nonempty collection of subsets of X.

(i) The pair $(X, \mathscr{S})$ is called a *choice space*.
(ii) A *choice function* (or *selection function*) on $(X, \mathscr{S})$ is a function $\gamma : \mathscr{S} \to \mathscr{P}(X)$ such that $\gamma(S) \subseteq S$ for every $S \in \mathscr{S}$.[1]

We call $S \in \mathscr{S}$ a *menu* (or *decision problem*) and $\gamma(S)$ a *choice set* (or the *admissible options*).

Definition 1.2. Let $(X, \mathscr{S})$ be a choice space.

(i) We say that $\mathscr{S}$ is *closed under arbitrary disjoint unions* if for every nonempty collection $I \subseteq \mathscr{S}$ of disjoint sets, $\bigcup_{S \in I} S \in \mathscr{S}$.

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[1] We do not require that $\mathscr{S}$ or $\gamma(\mathscr{S})$ consists solely of nonempty sets. This approach allows for more generality.

(ii) We say that $\mathscr{S}$ is *closed under nonempty relative complements* if whenever $S, T \in \mathscr{S}$ and $S \backslash T \neq \emptyset$, then $S \backslash T \in \mathscr{S}$.
(iii) We say that $\mathscr{S}$ *contains the unit* if $X \in \mathscr{S}$.
(iv) We say that $(X, \mathscr{S})$ is *closed under arbitrary disjoint unions* (*closed under nonempty relative complements, contains the unit*) if $\mathscr{S}$ is closed under arbitrary disjoint unions (closed under relative complements, contains the unit).

In the following, for a set X we let $\mathscr{P}_{fin}(X)$ denote the collection of all nonempty finite subsets of X.

Definition 1.3. Let γ be a choice function on a choice space $(X, \mathscr{S})$.
(i) We say that γ is *iterative* (with respect to $(X, \mathscr{S})$) if $\gamma(\mathscr{S}) \subseteq \mathscr{S}$, i.e., for every $S \in \mathscr{S}$, $\gamma(S) \in \mathscr{S}$.
(ii) We say that γ is *regular* if for every $S \in \mathscr{S}$, if $S \neq \emptyset$, then $\gamma(S) \neq \emptyset$.

In the study of rational choice, *coherence constraints* have been imposed on the form relationships may take among choices across varying menus. In other words, these requirements specify how choices must be made across different decision problems. We will focus on only a few of these coherence constraints:

(α) For every $S, T \in \mathscr{S}$, if $S \subseteq T$, then $S \cap \gamma(T) \subseteq \gamma(S)$. (*Sen's Property* α)
(β) For every $S, T \in \mathscr{S}$, if $S \subseteq T$ and $\gamma(T) \cap \gamma(S) \neq \emptyset$, then $\gamma(S) \subseteq \gamma(T)$. (*Sen's Property* β)
(Aiz) For every $S, T \in \mathscr{S}$, if $S \subseteq T$ and $\gamma(T) \subseteq S$, then $\gamma(S) \subseteq \gamma(T)$. (*Aizerman's Axiom*)

2. BINARY RELATIONS AND SZPILRAJN'S THEOREM

We now enumerate some properties of binary relations. We adopt the usual infix notation, writing xRy if $(x, y) \in R$. When there is no danger of confusion, we write $\neg xRy$ if $(x, y) \notin R$. Also, if R is a binary relation on X, we define R^{-1} by setting $R^{-1} := \{(x, y) : (y, x) \in R\}$.

Definition 2.1. Let X be a set, and let R be a binary relation on X.
(i) We call R *reflexive* if for every $x \in X$, xRx.
(ii) We call R *irreflexive* if for every $x \in X$, $\neg xRx$.
(iii) We call R *asymmetric* if for every $x, y \in X$, if xRy, then $\neg yRx$.
(iv) We call R *antisymmetric* if for every $x, y \in X$, if xRy and yRx, then $x = y$.
(v) We call R *complete* (or *total*) if for every $x, y \in X$, xRy or yRx.
(vi) We call R *connected* if for every $x, y \in X$, if $x \neq y$, then xRy or yRx.
(vii) We cal R *transitive* if for every $x, y, z \in X$, if xRy and yRz, then xRz.
(viii) We call R *modular* (or *negatively transitive*) if for every $x, y, z \in X$, if xRy, then xRz or zRy.
(ix) We call R *acyclic* if for every $n < \omega$ and $x_0, \ldots, x_{n-1} \in X$, if $x_i R x_{i+1}$ for each $i < n - 1$, then $\neg x_{n-1} R x_0$.

We now recall several composite properties of binary relations.

Definition 2.2. Let X be a set, and let R be a binary relation on X.
(i) We call R a *quasiorder* (or *preorder*) if it is reflexive and transitive.
(ii) We call R a *weak order* (or *total preorder*) if it is a complete quasiorder.
(iii) We call R a *partial order* if it an antisymmetric quasiorder.
(iv) We call R a *total order* (or *ordering*) if it is a complete partial order.
(v) We call R a *well-ordering* if it a total ordering such that for every nonempty $S \subseteq X$, there is $x \in S$ such that for all $y \in X$, xRy.

(vi) We call R a *strict weak order* if it is asymmetric and modular.
(vii) We call R a *strict partial order* if it is irreflexive and transitive.
(viii) We call R a *strict total order* (or *strict ordering*) if it is asymmetric, modular, and connected.

If R is a binary relation on X, we say that the pair (X, R) is a *quasiorder* (*weak order, partial order, etc.*) if R is a quasiorder (weak order, partial order, etc.). We invite the reader to investigate the interrelations among the foregoing properties.

Let us adopt the following terminology. If R_0 and R_1 are binary relations on X, call R_1 an *extension* of R_0 (with respect to X) if $R_0 \subseteq R_1$ and $R_0 \cap ((X \times X)\backslash R_0^{-1}) \subseteq R_1 \cap ((X \times X)\backslash R_1^{-1})$. Observe that if R_2 is an extension of R_1 and R_1 is an extension of R_0, then R_2 is an extension of R_0. Shortly we will show that every quasiorder has a weak order extension. We begin with a lemma.

Lemma 2.3. *Let $(X, \succeq)$ be a quasiorder, and let $v, w \in X$. Suppose that $v \not\succeq w$ and $w \not\succeq v$. Then $\succeq$ has a quasiorder extension $\gtrapprox$ such that $v \gtrapprox w$.*

Proof. Define a binary relation $\gtrapprox$ on X by setting

$$\gtrapprox := \succeq \cup \{(x, y) \in X \times X : x \succeq v \text{ and } w \succeq y\}.$$

Observe that since $\succeq$ is reflexive, we have that $v \gtrapprox w$. We claim that $\gtrapprox$ is a quasiorder extension of $\succeq$. We first establish that $\gtrapprox$ is a quasiorder. Since $\succeq \subseteq \gtrapprox$, it follows immediately that $\gtrapprox$ is reflexive. We show that $\gtrapprox$ is transitive. Let $x, y, z \in X$, and suppose $x \gtrapprox y$ and $y \gtrapprox z$. We consider four cases in turn.

(i) If $x \succeq y$ and $y \succeq z$, then $x \succeq z$ and so $x \gtrapprox z$.
(ii) If $x \succeq y$ and $y \succeq v$ and $w \succeq z$, then since $\succeq$ is transitive, it follows that $x \succeq v$, whereby $x \gtrapprox z$.
(iii) If $x \succeq v$ and $w \succeq y$ and $y \succeq z$, then since $\succeq$ is transitive, it follows that $w \succeq z$, whereby $x \gtrapprox z$.
(iv) If $x \succeq v$ and $w \succeq y$ and $y \succeq v$ and $w \succeq z$, then since $\succeq$ is transitive, it follows that $w \succeq v$, which is impossible.

Thus, $\gtrapprox$ is transitive, and $\gtrapprox$ is a quasiorder.

We now show that $\gtrapprox$ is an extension of $\succeq$. As indicated above, we have that $\succeq \subseteq \gtrapprox$. We establish that $\succeq \cap ((X \times X) \backslash \succeq^{-1}) \subseteq \gtrapprox \cap ((X \times X) \backslash \gtrapprox^{-1})$. Let $x, y \in X$, and suppose $x \succeq y$ and $y \not\succeq x$. Clearly $x \gtrapprox y$. If it were the case that $y \gtrapprox x$, then $y \succeq v$ and $w \succeq x$, so since $\succeq$ is transitive and since $w \succeq x$, $x \succeq y$, and $y \succeq v$, it would follow that $w \succeq v$, which is impossible. Thus, $x \not\gtrapprox y$. $\square$

Remark 2.4. Observe that by similar reasoning $\succeq$ has a quasiorder extension $\gtrapprox'$ such that $w \gtrapprox' v$. Thus, $\succeq$ has at least two quasiorder extensions, $\gtrapprox$ and $\gtrapprox'$, for which $v \gtrapprox w$ and $w \gtrapprox' v$.

Let us review the elements of Zorn's Lemma.

Definition 2.5. Let $(X, \leq)$ be a partial order.

(i) We call $\mathcal{C} \subseteq X$ a *chain* if for all $a, b \in \mathcal{C}$, either $a \leq b$ or $b \leq a$.
(ii) We say that $(X, \leq)$ is *inductive* if $X \neq \emptyset$ and for every nonempty chain $\mathcal{C} \subseteq X$, there is $a \in X$ such that for all $b \in \mathcal{C}$, $b \leq a$ (i.e., $\mathcal{C}$ is bounded above).
(iii) We call $a \in X$ a *maximal* member of $(X, \leq)$ if for every $b \in X$, if $a \leq b$, then $a = b$.

Theorem 2.6 (Zorn's Lemma)**.** *Every inductive partially ordered set has a maximal element.*

We now state and prove an important theorem based on a result due to Szpilrajn [8].[2]

Theorem 2.7. *Every quasiorder has a weak order extension.*

Proof. Let $(X, \geq)$ be a quasiorder. Set

$$\mathfrak{R} := \{\succsim \in \mathscr{P}(X \times X) : \succsim \text{ is a quasiorder extension of } \geq\}.$$

Observe that $(\mathfrak{R}, \subseteq)$ is a partial order. Also observe that $\geq \in \mathfrak{R}$. We show that $(\mathfrak{R}, \subseteq)$ is inductive. Let $\mathcal{C} \subseteq \mathfrak{R}$ be a nonempty chain, and set $\succeq := \bigcup_{\succsim \in \mathcal{C}} \succsim$. Clearly $\succsim \subseteq \succeq$ for every $\succsim \in \mathcal{C}$. We must accordingly show that $\succeq$ is a quasiorder extension of $\geq$.

We first establish that $(X, \succeq)$ is a quasiorder. Since $\mathcal{C}$ is nonempty and $\succsim$ is reflexive for every $\succsim \in \mathcal{C}$, it follows immediately that $\succeq$ is also reflexive. We claim that $\succeq$ is also transitive. Let $x, y, z \in X$, and suppose $x \succeq y$ and $y \succeq z$. Then there are $\succsim, \succsim' \in \mathcal{C}$ such that $x \succsim y$ and $y \succsim' z$. Since $\mathcal{C}$ is a chain, it follows that either $x \succsim' y$ or $y \succsim z$, whereby since $\succsim$ and $\succsim'$ are both transitive, either $x \succsim' z$ or $x \succsim z$, whence $x \succeq z$.

We now show that $\succeq$ is an extension of $\geq$. Since $\mathcal{C}$ is nonempty and $\geq \subseteq \succsim$ for each $\succsim \in \mathcal{C}$, we have that $\geq \subseteq \succeq$. Furthermore, $\geq \cap ((X \times X) \setminus \geq^{-1}) \subseteq \succeq \cap ((X \times X) \setminus \succeq^{-1})$, for if $x, y \in X$ are such that $x \geq y$ and $y \not\geq x$, then since $\succsim$ is an extension of $\geq$ for each $\succsim \in \mathcal{C}$, we have that $x \succsim y$ and $y \not\succsim x$ for each $\succsim \in \mathcal{C}$, so $x \succeq y$ and $y \not\succeq x$.

Hence, $\mathcal{C}$ is bounded above. Therefore, it follows by Zorn's Lemma that there is a maximal member $\succeq \in \mathfrak{R}$. We claim that $\succeq$ is complete. For *reductio ad absurdum*, assume there are $v, w \in X$ such that $v \not\succeq w$ and $w \not\succeq v$. Then by Lemma 2.3 it follows that $\succeq$ has a quasiorder extension $\succapprox$ such that $v \succapprox w$. But then $\succapprox$ is a quasiorder extension of $\geq$, $\succeq \subseteq \succapprox$, and $\succeq \neq \succapprox$, yielding a contradiction. ☐

The following corollary is an immediate consequence of Theorem 2.7.

Corollary 2.8. *Every transitive relation has a weak order extension.*

Proof. Let X be a set, and let R be a binary relation X. Let Δ be the diagonal of X, i.e., $\Delta := \{(x, x) : x \in X\}$. Define a binary relation $\geq$ on X by setting $\geq := \Delta \cup R$. Clearly $\geq$ is reflexive and transitive. By Theorem 2.7, $\geq$ has a weak order extension, which in turn is a weak order extension of R. ☐

We can also adapt the proofs of Lemma 2.3 and Theorem 2.7 to obtain Szpilrajn's Theorem [8]. As before, we begin with a lemma.

Lemma 2.9. *Let $(X, \succeq)$ be a partial order, and let $v, w \in X$. Suppose that $v \not\succeq w$ and $w \not\succeq v$. Then $\succeq$ has a partial order extension $\succapprox$ such that $v \succapprox w$.*

Proof. Define a binary relation $\succapprox$ as in the proof of Lemma 2.3. As before, we have that $v \succapprox w$. We must show that $\succapprox$ is a partial order extension of $\succeq$. In light of the proof of Lemma 2.3, we must only show that $\succapprox$ is antisymmetric. Let $x, y \in X$, and suppose $x \succapprox y$ and $y \succapprox x$. We consider four cases in turn.

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[2] Peter C. Fishburn in [2, pp. 16-18] shows that every strict partial order has a strict total order extension, while Bengt Hansson in [3, pp. 454-455] and Kotaro Suzumura in [7, pp. 15-16] show that every quasiorder has a weak order extension. Each proof invokes Zorn's Lemma. See Remark 2.11 below for a brief discussion of the proofs of Theorem 2.7 and Theorem 2.10 . Szpilrajn's original theorem asserts that every strict partial order has a strict total order extension, and his proof appeals to a version of Zorn's Lemma. As a historical aside, Szpilrajn remarks in his article that the "theorem is familiar, but the proofs, due to MM. Banach, Kuratowski, and Tarski, are not yet published" (see [6] for an English translation of [8]).

(i) If $x \succeq y$ and $y \succeq x$, then we immediately have that $x = y$.
(ii) If $x \succeq y$ and $y \succeq v$ and $w \succeq x$, then since $\succeq$ is transitive, it follows that $x \succeq v$, whereby $w \succeq v$, which is impossible.
(iii) If $x \succeq v$ and $w \succeq y$ and $y \succeq x$, then since $\succeq$ is transitive, it follows that $w \succeq x$, whereby $w \succeq v$, which is impossible.
(iv) If $x \succeq v$ and $w \succeq y$ and $y \succeq v$ and $w \succeq x$, then since $\succeq$ is transitive, it follows that $w \succeq v$, which is again impossible.

Thus, $\succapprox$ is antisymmetric, and $\succapprox$ is a partial order. One proceeds as in the proof of Lemma 2.3 to establish that $\succapprox$ is an extension of $\succeq$. Thus, $\succapprox$ is a partial order extension of $\succeq$. □

Corollary 2.10 (Szpilrajn's Theorem)**.**

(i) *Every partial order has a total order extension.*
(ii) *Every strict partial order has a strict total order extension.*

Proof. Observe that part (i) and part (ii) are equivalent. We prove part (i). Let $(X, \geq)$ be a partial order. Set

$$\mathfrak{R} := \{\succsim \in \mathscr{P}(X \times X) : \succsim \text{ is a partial order extension of } \geq\}.$$

Observe that $(\mathfrak{R}, \subseteq)$ is a partial order and that $\geq \in \mathfrak{R}$. As in the proof of Theorem 2.7, one must show that $(\mathfrak{R}, \subseteq)$ is inductive. Let $\mathcal{C} \subseteq \mathfrak{R}$ be a nonempty chain, and set $\succeq := \bigcup_{\succsim \in \mathcal{C}} \succsim$. Again, clearly $\succsim \subseteq \succeq$ for every $\succsim \in \mathcal{C}$. We must accordingly show that $\succeq$ is a partial order extension of $\geq$. To show that $\succeq$ is an extension of $\geq$, one proceeds as in the proof of Theorem 2.7. To verify that $\succeq$ is a partial order, it suffices to show that $\succeq$ is antisymmetric in light of the proof of Theorem 2.7,

Let $x, y \in X$, and suppose $x \succeq y$ and $y \succeq x$. Then there are $\succsim, \succsim' \in \mathcal{C}$ such that $x \succsim y$ and $y \succsim' x$. Since $\mathcal{C}$ is a chain, it follows that either $x \succsim' y$ or $y \succsim x$, whereby since $\succsim$ and $\succsim'$ are both antisymmetric, we have that $x = y$.

Hence, $\mathcal{C}$ is bounded above, so by Zorn's Lemma there is a maximal member $\succeq \in \mathfrak{R}$. As before, we must establish that $\succeq$ is complete. For *reductio ad absurdum*, assume that there are $v, w \in X$ such that $v \not\succeq w$ and $w \not\succeq v$. Then by Lemma 2.9 it follows that $\succeq$ has a partial order extension $\succapprox$ such that $v \succapprox w$. But then $\succapprox$ is a partial order extension of $\geq$, $\succeq \subseteq \succapprox$, and $\succeq \neq \succapprox$, yielding a contradiction. □

Remark 2.11. Theorem 2.7 and Szpilrajn's Theorem do not require the full strength of Zorn's Lemma (and so the Axiom of Choice). In fact, these results can be established using the Compactness Theorem of first-order logic (which in turn is equivalent to the Prime Ideal Theorem). Using the Compactness Theorem, however, creates no less work than that required in the proofs offered here. Indeed, one employs a similar argument (see [4] for a proof using a compactness argument).

Finally, we state and prove a theorem we will find useful later.

Theorem 2.12. *Let $(X, \succ)$ be a strict partial order, let $v \in X$, and let $S(v) := \{w \in X : v \neq w \text{ and } v \not\succ w \text{ and } w \not\succ v\}$. Then $\succ$ has a strict total order extension $>$ such for all $w \in S(v)$, $v > w$.*

Proof. Let $\kappa = |S(v)|$, and let $(w_\alpha : \alpha < \kappa)$ enumerate $S(v)$. We construct a sequence of binary relations $(\succ_\alpha \colon \alpha < \kappa)$ by transfinite recursion as follows. For each $\alpha < \kappa$, define

$$\succ_\alpha := \begin{cases} \succ \cup \{(x,y) \in X \times X : (x \succ v \text{ or } x = v) \text{ and } (w_0 \succ y \text{ or } w_0 = v)\} & \text{if } \alpha = 0 \\ \succ_\beta \cup \{(x,y) \in X \times X : (x \succ v \text{ or } x = v) \text{ and } (w_\alpha \succ y \text{ or } w_\alpha = y)\} & \text{if } \alpha = \beta + 1 \\ (\bigcup_{\beta<\alpha} \succ_\alpha) \cup \{(x,y) \in X \times X : (x \succ v \text{ or } x = v) \text{ and } (w_\alpha \succ y \text{ or } w_\alpha = y)\} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Observe that for all $\alpha, \beta < \kappa$, $\succ_\alpha$ is a strict partial order, $v \succ_\alpha w_\alpha$, and if $\alpha \leq \beta$, then $\succ_\alpha \subseteq \succ_\beta$. Set $>_0 := \bigcup_{\alpha<\kappa} \succ_\alpha$. Then $>_0$ is a strict partial order, so by Szpilrajn's Theorem $>_0$ has a strict total order extension $>$. Hence, by construction $>$ is a strict total order extension of $\succ$ and for all $w \in S(v)$, $v > w$.

$\square$

3. EXTENSIONS OF PSEUDO-RATIONALIZABILITY RESULTS

We recall a several definitions:

Definition 3.1. Let γ be a choice function on a choice space $(X, \mathscr{S})$. We say that a binary relation R on X (*transitive, complete, weak order, etc.*) *rationalizes* γ if for every $S \in \mathscr{S}$,

$$\gamma(S) = \{x \in S : xRy \text{ for all } y \in S\}.$$

We thereby call γ (*transitive, complete, weak order, etc.*) *rational* (or *rationalizable*) if there is a (transitive, complete, modular, etc.) binary relation R on X that rationalizes γ.

Definition 3.2. Let γ be a choice function on a choice space $(X, \mathscr{S})$.

(i) We say that γ is (*transitive, complete, weak order, etc.*) *G-rational* (or *G-rationalizable*) if there is a (transitive, complete, etc.) reflexive binary relation $\geq$ on X that rationalizes γ.

(ii) We say that γ is (*modular, acyclic, strict weak order, etc.*) *M-rational* (or *M-rationalizable*) if there is a (modular, acyclic, etc.) asymmetric binary relation $>$ on X such that $((X \times X)\setminus >)^{-1}$ rationalizes γ.

For our purposes, we recall the following well-known ("soundness") result:

Fact 3.3. *Let γ be a choice function on a choice space $(X, \mathscr{S})$. If γ is strict weak order M-rational (weak order G-rational), then γ satisfies condition α and condition β.*

Definition 3.4. Let γ be a choice function on a choice space $(X, \mathscr{S})$.

(i) We say that γ is (*weak order, total order, etc.*) *G-pseudo-rational* (or *G-pseudo-rationalizable*) *of order of at most κ* if there is a collection $(\gamma_i : i \in I)$ of (weak order, total order, etc.) regular G-rational choice functions on $(X, \mathscr{S})$ such that $|I| \leq \kappa$ and for all $S \in \mathscr{S}$,

$$\gamma(S) = \bigcup_{i \in I} \gamma_i(S)$$

We thereby say that $(\gamma_i : i \in I)$ *G-pseudo-rationalizes* γ.

(ii) We say that γ is (*strict weak order, strict total order, etc.*) *M-pseudo-rational* (or *M-pseudo-rationalizable*) *of order of at most κ* if there is a collection $(\gamma_i : i \in I)$ of (strict weak order, strict total order, etc.) regular M-rational choice functions on $(X, \mathscr{S})$ such that $|I| \leq \kappa$ and for all $S \in \mathscr{S}$,

$$\gamma(S) = \bigcup_{i \in I} \gamma_i(S)$$

We thereby say that $(\gamma_i : i \in I)$ *M-pseudo-rationalizes* γ.

Theorem 3.5. *Let γ be a regular, iterative choice function on a choice space $(X, \mathscr{S})$ which contains the unit and which is closed under arbitrary disjoint unions and nonempty relative complements. Let $\kappa := |X|$, and let $\lambda := |\mathscr{S}|$. Then γ is strict weak order M-pseudo-rationalizable of order at most $\lambda \leq 2^\kappa$ if and only if γ satisfies condition α and condition* AA.

Proof.

(⇒) Suppose γ is M-pseudo-rationalizable (of order at most λ), and let $(\gamma_i : i \in I)$ M-pseudo-rationalize γ.

Condition α: Let $S, T \in \mathscr{S}$, and suppose $S \subseteq T$. Assume $x \in S \cap \gamma(T)$. Then for some $i \in I$, $x \in \gamma_i(T)$ and since γ_i satisfies condition α, we have $S \cap \gamma_i(T) \subseteq \gamma_i(S)$ and so $x \in \gamma_i(S)$.

Condition Aiz: Let $S, T \in \mathscr{S}$, and suppose $S \subseteq T$ and $\gamma(T) \subseteq S$. We may assume $T \neq \emptyset$. Then since γ_i is regular for each $i \in T$, $\gamma_i(T) \neq \emptyset$ for each $i \in I$. Therefore, since γ_i satisfies condition α and $\gamma_i(T) \subseteq S$ for each $i \in I$, we have that $\gamma_i(T) = \gamma_i(T) \cap S \subseteq \gamma_i(S)$ and so $\gamma_i(T) \cap \gamma_i(S) \neq \emptyset$ for each $i \in I$. Now since γ_i satisfies condition β for each $i \in I$, it follows that $\gamma_i(S) \subseteq \gamma_i(T)$ for each $i \in I$. Therefore, we have that $\gamma(S) = \bigcup_{i \in I} \gamma_i(S) \subseteq \bigcup_{i \in I} \gamma_i(T) = \gamma(T)$, as desired.

(⇐) Suppose γ satisfies condition α and condition Aiz.

Let $S \in \mathscr{S}$ be a nonempty set. We construct a sequence of sets $(X_\alpha : \alpha < \xi)$ by transfinite recursion as follows. Assuming X_β has been defined for all $\beta < \delta$, define

$$X_\delta := \begin{cases} \gamma(X \backslash \bigcup_{\beta<\delta} X_\beta) & \text{if } X \backslash \bigcup_{\beta<\delta} X_\beta \neq \emptyset \text{ and } \gamma(X \backslash \bigcup_{\beta<\delta} X_\beta) \subseteq S \\ \gamma(X \backslash \bigcup_{\beta<\delta} X_\beta) \backslash S & \text{if } X \backslash \bigcup_{\beta<\delta} X_\beta \neq \emptyset \text{ and } \gamma(X \backslash \bigcup_{\beta<\delta} X_\beta) \nsubseteq S \\ \emptyset & \text{otherwise.} \end{cases}$$

It is a simple matter to verify that there is a least $\eta \in \mathbf{ON}$ for which $X_\eta \subseteq S$, and $S \subseteq X \backslash \bigcup_{\beta<\eta} X_\beta$. Thus, by condition α and condition Aiz it follows that $X_\eta = \gamma(X \backslash \bigcup_{\beta<\eta} X_\beta) = \gamma(S)$. It is equally easy to check that there is a least $\xi \in \mathbf{ON}$ such that $X_\xi = \emptyset$ and $\eta < \xi$.

Observe that $(X_\alpha : \alpha < \xi)$ is a partition of X. Define a binary relation $>$ on X by setting

$$> := \bigcup_{\alpha, \beta < \xi : \alpha < \beta} X_\alpha \times X_\beta.$$

Clearly $>$ thus defined is asymmetric. Observe that $>$ is also modular:

> Let $x, y, z \in X$. Suppose $x > y$ and $x \not> z$. Then there are $\beta, \zeta < \xi$ such that $\beta < \zeta$ and $x \in X_\beta$ and $y \in X_\zeta$, and for all $\delta < \xi$ such that $\beta < \delta$, $z \notin X_\delta$. Since $(X_\alpha : \alpha < \xi)$ is a partition of X, it follows that $z \in X_\delta$ for some $\delta < \xi$, so $\delta \leq \beta < \zeta$ and therefore $z > y$.

Define a choice function γ_S on $(X, \mathscr{S})$ by setting for all $T \in \mathscr{S}$,

$$\gamma_S(T) := \{x \in T : y > x \text{ for no } y \in T\}.$$

We claim that the following properties are satisfied:

(i) γ_S is a regular modular M-rational choice function.
(ii) For every $T \in \mathscr{S}$, $\gamma_S(T) \subseteq \gamma(T)$.
(iii) $\gamma(S) \subseteq \gamma_S(S)$.

We establish these claims in order.

(i) Clearly γ_S is a modular M-rational choice function. Moreover, since there are no infinite descending chains of ordinals, γ_S is regular.

(ii) Let $T \in \mathscr{S}$. We may assume that $T \neq \emptyset$, so $\gamma_S(T) \neq \emptyset$. Then since $(X_\alpha : \alpha < \xi)$ partitions X, there is $\delta < \xi$ such that $\gamma_S(T) \cap X_\delta \neq \emptyset$. Choose $w \in \gamma_S(T) \cap X_\delta$. We claim that $\gamma_S(T) \subseteq X_\delta$.

Let $x \in \gamma_S(T)$. For *reductio ad absurdum,* assume that there is $\beta < \xi$ for which $\delta \neq \beta$ and $x \in \gamma_S(T) \cap X_\beta$. Then since $\delta < \beta$ or $\beta < \delta$, it must be the case that $w > x$ or $x > w$, so $x \notin \gamma_S(T)$ or $w \notin \gamma_S(T)$, yielding a contradiction. Hence, since $(X_\alpha : \alpha < \xi)$ partitions X, it follows that $x \in X_\delta$, as desired.

Now since $T \subseteq X \backslash \bigcup_{\beta<\delta} X_\beta$, by condition α it follows that $\gamma(X \backslash \bigcup_{\beta<\delta} X_\beta) \cap T \subseteq \gamma(T)$. On the one hand, if $X_\delta = \gamma(X \backslash \bigcup_{\beta<\delta} X_\beta) \backslash S$, then

$$\begin{aligned} \gamma_S(T) &\subseteq X_\delta \cap T \\ &= (\gamma(X \backslash \bigcup_{\beta<\delta} X_\beta) \backslash S) \cap T \\ &\subseteq \gamma(X \backslash \bigcup_{\beta<\delta} X_\beta) \cap T \\ &\subseteq \gamma(T). \end{aligned}$$

On the other hand, if $X_\delta = \gamma(X \backslash \bigcup_{\beta<\delta} X_\delta)$, then

$$\begin{aligned} \gamma_S(T) &\subseteq X_\delta \cap T \\ &= \gamma(X \backslash \bigcup_{\beta<\delta} X_\delta) \cap T \\ &\subseteq \gamma(T). \end{aligned}$$

Thus, $\gamma_S(T) \subseteq \gamma(T)$.

(iii) Let $x \in \gamma(S)$. For *reductio ad absurdum*, assume that $x \notin \gamma_S(S)$. Then there is $y \in S$ such that $y > x$. Since $\gamma(S) = X_\eta$, there must be some $\alpha < \eta$ for which $y \in X_\alpha$. But by construction $X_\alpha = \gamma(X \backslash \bigcup_{\beta<\alpha} X_\beta) \backslash S$, so $y \notin S$, yielding a contradiction. Hence, $x \in \gamma_S(S)$ and so $\gamma(S) \subseteq \gamma_S(S)$.

We have shown that for every $S \in \mathscr{S}$, there is a regular modular M-rational choice function γ_S such that for every $T \in \mathscr{S}$, $\gamma_S(T) \subseteq \gamma(T)$. (Note that if $S = \emptyset$, we may let $\gamma_S := \gamma_X$.) Thus, we have that for every $S \in \mathscr{S}$,

$$\bigcup_{T \in \mathscr{S}} \gamma_T(S) \subseteq \gamma(S),$$

Furthermore, since we have shown that for each $S \in \mathscr{S}$, $\gamma(S) = \gamma_S(S)$, it follows that

$$\gamma(S) = \bigcup_{T \in \mathscr{S}} \gamma_T(S).$$

Finally, because we have defined a choice function γ_S for each $S \in \mathscr{S}$, γ is pseudo-rationalizable of order at most $\lambda \leq 2^\kappa$.

$\square$

Remark 3.6. Observe that if for each $S \in \mathscr{S}$ we define $\geq := \{(x, y) : y \not> x\}$, then $\geq$ is complete and transitive, and $\geq$ G-rationalizes γ_S. We therefore immediately have the following additional result.

Corollary 3.7. *Let γ be a regular, iterative choice function on a choice space $(X, \mathscr{S})$ which contains the unit and which is closed under arbitrary disjoint unions and nonempty relative complements. Let $\kappa := |X|$, and let $\lambda := |\mathscr{S}|$. Then γ is weak order G-pseudo-rationalizable of order at most $\lambda \leq 2^\kappa$ if and only if γ satisfies condition α and condition* AA.

Here we present a simple example illustrating the general idea behind the procedure involved in the foregoing proof.

Example 3.8. Consider a choice function γ on a choice space $(X, \mathscr{P}_{fin}(X))$, where $X := \{x, y, z\}$ and

$$\gamma(\{x\}) := \{x\},\ \gamma(\{y\}) := \{y\},\ \gamma(\{z\}) := \{z\},$$
$$\gamma(\{x, y\}) := \{x, y\},\ \gamma(\{x, z\}) := \{x, z\},\ \gamma(\{y, z\}) := \{y, z\}, \text{ and}$$
$$\gamma(\{x, y, z\}) := \{x, y\}.$$

It is an easy matter to check that γ thus defined satisfies condition α and condition AA. Observe that γ violates condition γ, so γ is not rationalizable.

We now follow the procedure outlined in the $\Leftarrow$-direction of the proof of Theorem 3.5. This procedure guarantees a strict weak order $>_S$ for each $S \in \mathscr{P}_{fin}(X)$. Let us first consider the case in which $S := \{x\}$. Then:

$X_0 = \gamma(X) \backslash S = \{y\}$.
$X_1 = \gamma(X \backslash X_0) \backslash S = \{z\}$.
$X_2 = \gamma(X \backslash (X_0 \cup X_1)) = \{x\}$.
$X_\alpha = \emptyset$ for all $\alpha > 2$.

This generates a binary relation $>_{\{x\}}$ for which $y >_{\{x\}} z >_{\{x\}} x$ and $y >_{\{x\}} x$. Similarly, for $S := \{y\}$, the procedure generates a binary relation $>_{\{y\}}$ for which $x >_{\{y\}} z >_{\{y\}} y$ and $x >_{\{y\}} y$. Observe that in these cases both $>_{\{x\}}$ and $>_{\{y\}}$ are strict total orders. Now consider the case in which $S := \{z\}$:

$X_0 = \gamma(X) \backslash S = \{x, y\}$.
$X_1 = \gamma(X \backslash X_0) = \{z\}$.
$X_\alpha = \emptyset$ for all $\alpha > 1$.

We thereby obtain a strict weak order $>_{\{z\}}$ according to which $x >_{\{z\}} z$ and $y >_{\{z\}} z$. Let us repeat the procedure for the remainder of the $S \in \mathscr{P}_{fin}(X)$.

$S := \{x, y\}$.
$X_0 = \gamma(X) = \{x, y\}$.
$X_1 = \gamma(X \backslash X_0) \backslash S = \{z\}$
$X_\alpha = \emptyset$ for all $\alpha > 1$.

$S := \{x, z\}$.
$X_0 = \gamma(X) \backslash S = \{y\}$.
$X_1 = \gamma(X \backslash X_0) = \{x, z\}$.
$X_\alpha = \emptyset$ for all $\alpha > 1$.

$S := \{y, z\}$.

$X_0 = \gamma(X)\backslash S = \{x\}$.
$X_1 = \gamma(X\backslash X_0) = \{y, z\}$.
$X_\alpha = \emptyset$ for all $\alpha > 1$.

$S := X$.
$X_0 = \gamma(X) = \{x, y\}$.
$X_1 = \gamma(X\backslash X_0) = \{z\}$.
$X_\alpha = \emptyset$ for all $\alpha > 1$.

We then define $>_{\{x,y\}}$, $>_{\{x,z\}}$, $>_{\{y,z\}}$, and $>_X$ as before. Observe that $>_X = >_{\{x,y\}} = >_{\{z\}}$. Thus, the procedure generates five strict weak orders. However, only two strict weak orders are needed to pseudo-rationalize γ, viz., $>_{\{x,z\}}$ and $>_{\{y,z\}}$. Observe that there are 13 possible strict weak orders on X.

Whereas the previous theorem requires the existence of a collection of strict weak orders, the next theorem requires the existence of strict total orders.

Theorem 3.9. *Let γ be a regular choice function on a choice space $(X, \mathscr{P}(X))$. Let $\kappa := |X|$, and suppose $\kappa \geq \aleph_0$. Then γ is strict total order M-pseudo-rationalizable of order at most $\sum_{S\in\mathscr{P}(X)} |\gamma(S)| \leq 2^\kappa$ if and only if γ satisfies condition α and condition* AA.

Proof.

($\Rightarrow$) Proceed as in Theorem 3.5.

($\Leftarrow$) Suppose γ satisfies condition α and condition Aiz.

Let $S \in \mathscr{P}(X)$ be a nonempty set, let $v \in \gamma(S)$, and let $(X, \preceq)$ be a well-ordering for which v is the least element of X. Construct a sequence $(x_\alpha : \alpha < \xi)$ by transfinite recursion as follows. Assuming x_β has been defined for all $\beta < \delta$, define

$$x_\delta := \begin{cases} \min_\preceq(\gamma(X\backslash\{x_\beta : \beta < \delta\})) & \text{if } X\backslash\{x_\beta : \beta < \delta\} \neq \emptyset \text{ and } \gamma(X\backslash\{x_\beta : \beta < \delta\}) \subseteq S \\ \min_\preceq(\gamma(X\backslash\{x_\beta : \beta < \delta\})\backslash S) & \text{if } X\backslash\{x_\beta : \beta < \delta\} \neq \emptyset \text{ and } \gamma(X\backslash\{x_\beta : \beta < \delta\}) \nsubseteq S \\ \emptyset & \text{otherwise.} \end{cases}$$

It is a simple matter to verify that there is a least $\eta \in \mathbf{ON}$ for which $\gamma(X\backslash\{x_\beta : \beta < \eta\}) \subseteq S$ and $S \subseteq X\backslash\{x_\beta : \beta < \eta\}$. Thus, by condition α and condition Aiz it follows that $\gamma(X\backslash\{x_\beta : \beta < \eta\}) = \gamma(S)$. Since $v \in \gamma(S)$, we have that $v = \min_\preceq(\gamma(X\backslash\{x_\beta : \beta < \eta\})) = x_\eta$. It is equally easy to check that there is a least $\xi \in \mathbf{ON}$ such that $x_\xi = \emptyset$ and $\eta < \xi$.

Observe that $(x_\alpha : \alpha < \xi)$ exhausts X. Define a binary relation $>_{S,v}$ on X by setting

$$>_{S,v} := \{(x, y) \in X \times X : \text{there are } \alpha, \beta < \xi \text{ such that } \alpha < \beta \text{ and } x = x_\alpha \text{ and } y = x_\beta\}.$$

Clearly $>_{S,v}$ thus defined is asymmetric and connected. Observe that $>_{S,v}$ is also modular:

Let $x, y, z \in X$. Suppose $x >_{S,v} y$ and $x \not>_{S,v} z$. Then there are $\beta, \zeta < \xi$ such that $\beta < \zeta$ and $x = x_\beta$ and $y = x_\zeta$, and for all $\delta < \xi$ such that $\beta < \delta$, $z \neq x_\delta$. Since $(x_\alpha : \alpha < \xi)$ exhausts X, it follows that $z = x_\delta$ for some $\delta < \xi$, so $\delta \leq \beta < \zeta$ and therefore $z >_{S,v} y$.

Define a choice function $\gamma_{S,v}$ on $(X, \mathscr{P}(X))$ by setting for all $T \in \mathscr{P}(X)$,

$$\gamma_{S,v}(T) := \{x \in T : y >_{S,v} x \text{ for no } y \in T\}.$$

We claim that the following properties are satisfied:

(i) $\gamma_{S,v}$ is a regular strict total order M-rational choice function.
(ii) For every $T \in \mathscr{P}(X)$, $\gamma_{S,v}(T) \subseteq \gamma(T)$.
(iii) $v \in \gamma_{S,v}(S)$.

We establish these claims in order.

(i) Clearly $\gamma_{S,v}$ is a strict total order M-rational choice function. Moreover, since there are no infinite descending chains of ordinals, $\gamma_{S,v}$ is regular. Observe that since $>_{S,v}$ is a strict total order, for all $T \in \mathscr{P}(X)$, if $T \neq \emptyset$, then $|\gamma_{s,v}(T)| = 1$.

(ii) Let $T \in \mathscr{P}(X)$. We may assume that $T \neq \emptyset$, so $\gamma_{S,v}(T) \neq \emptyset$. Then since $(x_\alpha : \alpha < \xi)$ exhausts X, there is $\delta < \xi$ such that $\{x_\delta\} = \gamma_{S,v}(T)$. Since $T \subseteq X \backslash \{x_\beta : \beta < \delta\}$, by condition α it follows that $\gamma(X \backslash \{x_\beta : \beta < \delta\}) \cap T \subseteq \gamma(T)$. On the one hand, if $x_\delta = \min_{\preceq}(\gamma(X \backslash \{x_\beta : \beta < \delta\}) \backslash S)$, then $x_\delta \in (\gamma(X \backslash \{x_\beta : \beta < \delta\}) \backslash S) \cap T \subseteq \gamma(X \backslash \{x_\beta : \beta < \delta\}) \cap T \subseteq \gamma(T)$, so $x_\delta \in \gamma(T)$. On the other hand, if $x_\delta = \min_{\preceq}(\gamma(X \backslash \{x_\beta : \beta < \delta\}))$, then $x_\delta \in \gamma(X \backslash \{x_\beta : \beta < \delta\}) \cap T \subseteq \gamma(T)$, so $x_\delta \in \gamma(T)$. Thus, $\gamma_{S,v}(T) \subseteq \gamma(T)$.

(iii) For *reductio ad absurdum*, assume that $v \notin \gamma_{S,v}(S)$. Then there is $y \in S$ such that $y >_{S,v} v$. Then there must be some $\alpha < \eta$ for which $y = x_\alpha$. But by construction $x_\alpha \in \gamma(X \backslash \{x_\beta : \beta < \delta\}) \backslash S$, so $y \notin S$, yielding a contradiction. Hence, $v \in \gamma_{S,v}(S)$.

We have shown that for every $S \in \mathscr{P}(X)$ and $v \in \gamma(S)$, there is a regular strict total order M-rational choice function $\gamma_{S,v}$ such that for every $T \in \mathscr{P}(X)$, $\gamma_{S,v}(T) \subseteq \gamma(T)$. (Note that if $S = \emptyset$, we may let $\gamma_{S,v} := \gamma_{X,w}$ for some chosen $w \in \gamma(X)$.) Thus, we have that for every $S \in \mathscr{P}(X)$,

$$\bigcup_{T \in \mathscr{P}(X),\, v \in \gamma(T)} \gamma_{T,v}(S) \subseteq \gamma(S),$$

Furthermore, since we have shown that for each $S \in \mathscr{P}(X)$ and $v \in \gamma(S)$ we have $v \in \gamma_{S,v}(S)$, it follows that

$$\gamma(S) = \bigcup_{T \in \mathscr{P}(X),\, v \in \gamma(T)} \gamma_{T,v}(S).$$

□

The following example illustrates the procedure outlined in the proof of Theorem 3.9.

Example 3.10. Recall the choice function γ on the choice space $(X, \mathscr{P}_{fin}(X))$ of Example 3.8. There γ was the identity function on every $S \in \mathscr{P}_{fin}(X)$ except X, in which case $\gamma(X) = \{x, y\}$.

According to the procedure described in the $\Leftarrow$-direction of the proof of Theorem 3.9, we construct a strict total order $>_{S,v}$ for every $S \in \mathscr{P}_{fin}(X)$ and $v \in \gamma(S)$. Let us first consider the case in which $S := \{x\}$. In this case $\gamma(S) = \{x\}$, so we must construct only one strict total order. Choose a well-ordering $\preceq_{\{x\},x}$ of X for which x is the least element of X, say, $x \prec_{\{x\},x} y \prec_{\{x\},x} z$ and $x \prec_{\{x\},x} z$. Then:

$x_0 = \min_{\preceq_{\{x\},x}}(\gamma(X)\backslash S) = \{y\}$.
$x_1 = \min_{\preceq_{\{x\},x}}(\gamma(X\backslash\{x_0\})\backslash S) = \{z\}$.
$x_2 = \min_{\preceq_{\{x\},x}}(\gamma(X\backslash\{x_0, x_1\})) = \{x\}$.
$x_\alpha = \emptyset$ for all $\alpha > 2$.

We accordingly obtain a strict total order $>_{\{x\},x}$ for which $y >_{\{x\},x} z >_{\{x\},x} x$, and $y >_{\{x\},x} x$. For $S := \{y\}$, we choose a well-ordering $\preceq_{\{y\},y}$ of X for which y is the least element of X, say, $y \prec_{\{y\},y} z \prec_{\{y\},y} x$ and $y \prec_{\{y\},y} x$. The procedure then generates a strict total order $>_{\{y\},y}$ for which $x >_{\{y\},y} z >_{\{y\},y} y$ and $x >_{\{y\}} z$.

Let us consider the case in which $S := X$. In this case, we must construct strict total orders for x and y. To construct $>_{X,x}$, choose the well-ordering $\preceq_{X,x}$ of X for which $x \prec_{X,x} y \prec_{X,x} z$ and $x \prec_{X,x} z$. Then:

$x_0 = \min_{\preceq_{X,x}}(\gamma(X)) = \{x\}$.
$x_1 = \min_{\preceq_{X,x}}(\gamma(X\backslash\{x_0\})) = \{y\}$.
$x_2 = \min_{\preceq_{X,x}}(\gamma(X\backslash\{x_0, x_1\})) = \{z\}$.
$x_\alpha = \emptyset$ for all $\alpha > 2$.

We thereby obtain a strict total order $>_{X,x}$ according to which $x >_{X,x} y >_{X,x} z$ and $x >_{X,x} z$. As for $>_{X,y}$, choose the well-ordering $\preceq_{X,y}$ of X for which $y \prec_{X,y} x \prec_{X,y} z$ and $y \prec_{X,y} z$. Then:

$x_0 = \min_{\preceq_{X,y}}(\gamma(X)) = \{y\}$.
$x_1 = \min_{\preceq_{X,y}}(\gamma(X\backslash\{x_0\})) = \{x\}$.
$x_2 = \min_{\preceq_{X,y}}(\gamma(X\backslash\{x_0, x_1\})) = \{z\}$.
$x_\alpha = \emptyset$ for all $\alpha > 2$.

Thus, $y >_{X,y} x >_{X,y} z$ and $y >_{X,y} z$. Observe that the outcome of the procedure depends on the choice of the well-ordering of X. For example, if $\preceq'_{X,y}$ were chosen instead of $\preceq_{X,y}$, where $y \prec'_{X,y} z \prec'_{X,y} x$ and $y \prec'_{X,y} x$, then we would have arrived at a strict total order $>'_{X,y}$ for which $y >'_{X,y} z >'_{X,y} x$ and $y >'_{X,y} x$. In fact, given a certain collection of well-orders over X — one for each $S \in \mathscr{P}_{fin}(X)$ and $v \in \gamma(S)$ — one can produce four out of the six possible strict total orders on X. Of course, the two excluded strict totals orders are those for which z is ranked highest. However, only two strict total orders are required to pseudo-rationalize γ, viz., $>_{X,x}$ and $>_{X,y}$.

Theorem 3.11. *Let γ be a regular choice function on a choice space $(X, \mathscr{P}_{fin}(X))$ for which $|X| = \aleph_0$. Then γ is strict total order M-pseudo-rationalizable of order at most $\sum_{S\in\mathscr{P}_{fin}(X)} |\gamma(S)| \leq \aleph_0$ if and only if γ satisfies condition α and condition* AA.

Proof.

($\Rightarrow$) Proceed as in the previous theorems.

($\Leftarrow$) Suppose γ satisfies condition α and condition AA. Let $(x_n : n < \omega)$ be a enumeration of X, and for each $n < \omega$, let $X_n := (x_i : i < n)$. Let $S \in \mathscr{S}$ be a nonempty set, and let n_0 be the least natural number for which $S \subseteq X_{n_0}$. By the proof of Theorem 3.5, for each $n < \omega$, if $n \geq n_0$, there is a strict weak order $>_n$ and a choice function γ_n on $(X_n, \mathscr{P}(X_n))$ satisfying the following properties:

(i) For every $T \in \mathscr{P}(X_n)$, $\gamma_n(T) = \{x \in T : y >_n x \text{ for no } y \in T\}$.
(ii) γ_n is a regular strict weak order M-rational choice function.
(iii) For every $T \in \mathscr{P}(X_n)$, $\gamma_n(T) \subseteq \gamma(T)$.
(iv) $\gamma_n(S) \subseteq \gamma(S)$.

For each $n < \omega$, define $u_n : X \to [0, 2]$ by setting

$$u_n(x_m) := \sum_{i<n \text{ and } x_m >_n x_i} \frac{1}{2^i}$$

for each $m < \omega$ such that $m < n$ and $n \geq n_0$, and $u_n(x_m) := 0$ if $m \geq n$ or $n < n_0$. Observe that for every $l, m, n < \omega$, if $m, l < n$ and $n \geq n_0$, then

$$x_m >_n x_l \quad \text{if and only if} \quad u_n(x_m) > u_n(x_l).$$

Now define $u : X \to [0, 2]$ by setting for all $x \in X$,

$$u(x) := \limsup_{n\to\infty} u_n(x).$$

Observe that for each $x \in X$, $\limsup_{n\to\infty} u_n(x) \in [0, 2]$. Finally, define a binary relation $\succ_S$ on X by setting for every $x, y \in X$,

$$x \succ_S y \quad : \text{if and only if} \quad u(x) > u(y).$$

Clearly $\succ_S$ thus defined is asymmetric and modular, i.e., a strict weak order. Define a choice function γ_S on $(X, \mathscr{P}_{fin}(X))$ by setting for all $T \in \mathscr{P}_{fin}(X)$,

$$\gamma_S(T) := \{x \in T : y \succ_S x \text{ for no } y \in T\}.$$

We claim that the following properties are satisfied:

(v) γ_S is a regular strict weak order M-rational choice function.
(vi) For every $T \in \mathscr{P}_{fin}(X)$, $\gamma_S(T) \subseteq \gamma(T)$.
(vii) $\gamma(S) \subseteq \gamma_S(S)$.

As usual, we establish these claims in order.

(v) Since $\mathscr{P}_{fin}(X)$ consists solely of all nonempty finite subsets of X and $\succ_S$ is a strict weak order (and so acyclic), it follows that $\gamma_S(T) \neq \emptyset$ for each $T \in \mathscr{P}_{fin}(X)$.

(vi) Let $T \in \mathscr{P}_{fin}(X)$, and let $x_m \in T$. Suppose $x_m \notin \gamma(T)$. Let n_1 be the least natural number for which $S \cup T \subseteq X_{n_1}$. Let $n < \omega$ be such that $n \geq n_1$. Then by (iii) we have $x_m \notin \gamma_n(T)$ and so there is $y \in T$ such that $y >_n x_m$, whereby $u_n(y) > u_n(x_m)$. Therefore, for every $n < \omega$ such that $n \geq n_1$, there is $y \in T$ such that

$$\begin{aligned} \max(u_n(z) : z \in T) &\geq u_n(y) \\ &\geq u_n(x_m) + \frac{1}{2^m} \\ &> u_n(x_m). \end{aligned}$$

Hence,

$$\begin{aligned} \limsup_{n\to\infty} \max(u_n(z) : z \in T) &\geq \limsup_{n\to\infty} (u_n(x_m) + \frac{1}{2^m}) \\ &> \limsup_{n\to\infty} u_n(x_m). \end{aligned}$$

Now since T is finite, it follows that

$$\begin{aligned}
\max(u(z) : z \in T) &= \max(\limsup_{n\to\infty} u_n(z) : z \in T) \\
&= \limsup_{n\to\infty} \max(u_n(z) : z \in T) \\
&\geq \limsup_{n\to\infty} (u_n(x_m) + \frac{1}{2^m}) \\
&> \limsup_{n\to\infty} u_n(x_m) \\
&= u(x_m).
\end{aligned}$$

Thus, there is $z \in T$ such that $u(z) > u(x_m)$, whence $z \succ_S x_m$ and so $x_m \notin \gamma_S(T)$. Hence, $\gamma_S(T) \subseteq \gamma(T)$.

(vii) Let $x \in \gamma(S)$. For *reductio ad absurdum*, assume that $x \notin \gamma_S(S)$. Then there is $y \in S$ such that $y \succ_S x$ and so $u(y) > u(x)$. But then $\limsup_{n\to\infty} u_n(y) > \limsup_{n\to\infty} u_n(x)$, so there is $n < \omega$ such that $u_n(y) > u_n(x)$. Since $u_n(y) > 0$, by construction of u_n it follows that $n \geq n_0$ and $x, y \in X_n$. But then $y >_n x$ and so $x \notin \gamma_n(S)$, contradicting (iv). Hence, $x \in \gamma_{S,v}(S)$.

Thus, conditions (v), (vi), and (vii) are satisfied. Now let $v \in \gamma(S)$. Observe that $(X, \succ_S)$ is a strict partial order. Let $S(v) := \{w \in X : v \neq w \text{ and } v \not\succ_S w \text{ and } w \not\succ_S v\}$. Then it follows by Theorem 2.12 that $\succ_S$ has a strict total order extension $>_{S,v}$ such that for every $w \in S(v)$, $v >_{S,v} w$.

Define a choice function $\gamma_{S,v}$ on $(X, \mathscr{P}_{fin}(X))$ by setting for all $T \in \mathscr{P}_{fin}(X)$,

$$\gamma_{S,v}(T) := \{x \in T : y >_{S,v} x \text{ for no } y \in T\}.$$

As before, we claim that the following properties are satisfied:

(viii) $\gamma_{S,v}$ is a regular strict total order M-rational choice function.
(ix) For every $T \in \mathscr{P}_{fin}(X)$, $\gamma_{S,v}(T) \subseteq \gamma(T)$.
(x) $\gamma_{S,v}(S) = \{v\}$.

As for (viii), since each $T \in \mathscr{P}_{fin}(X)$ is finite and $>_{S,v}$ is a strict total order (and so acyclic), it follows that condition (viii) holds. For (ix), observe that for every $T \in \mathscr{P}_{fin}(X)$ such that $x \in T$, if for all $y \in T$, $y \not>_{S,v} x$, then since $>_{S,v}$ is an extension of $\succ_S$, we have that for all $y \in T$, $y \not\succ_S x$. Thus, condition (ix) is satisfied.

Now for *reductio ad absurdum*, assume that $v \notin \gamma_{S,v}(S)$. Then there is $w \in S$ such that $w >_{S,v} v$. Since $>_{S,v}$ is asymmetric, we have that $v \not>_{S,v} w$, so since $>_{S,v}$ is an extension of $\succ_S$, it follows that $v \not\succ_S w$. Observe that since $v \in \gamma_S(S)$, we have that $w \not\succ_S v$. But then $v \neq w$, $v \not\succ_S w$, and $w \not\succ_S v$, so $w \in S(v)$, whence $v >_{S,v} w$, yielding a contradiction.

We have thereby shown that for every $S \in \mathscr{P}_{fin}(X)$ and $v \in \gamma(S)$, there is a regular strict total order M-rational choice function $\gamma_{S,v}$ such that for every $T \in \mathscr{P}_{fin}(X)$, $\gamma_{S,v}(T) \subseteq \gamma(T)$. Thus, we have that for every $S \in \mathscr{P}_{fin}(X)$,

$$\bigcup_{T \in \mathscr{P}_{fin}(X),\, v \in \gamma(T)} \gamma_{T,v}(S) \subseteq \gamma(S),$$

Furthermore, since we have shown that for each $S \in \mathscr{P}_{fin}(X)$, $\gamma(S) \subseteq \gamma_{S,v}(S)$, it follows that

$$\gamma(S) = \bigcup_{T \in \mathscr{P}_{fin}(X),\, v \in \gamma(T)} \gamma_{T,v}(S).$$

□

In light of the proof of Theorem 3.11, we also have the following result.

Corollary 3.12. *Let γ be a regular choice function on a choice space $(X, \mathscr{P}_{fin}(X))$ for which $|X| = \aleph_0$. Then γ is strict weak order M-pseudo-rationalizable of order at most $\aleph_0$ if and only if γ satisfies condition α and condition* AA.

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