Engineering Design I: Methods and Skills
Topic Readings

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Chapter 9

Motors: Modeling and Selection

9.1 Electric Motors

An electric motor converts electrical energy and current into mechanical energy and torque. Electric motors are prevalent in robotics and consumer electronics applications, often in combination with an energy-dense chemical battery.

Basic operating principle of a motor:
When a conductor with an electrical current is located in a magnetic field, the conductor experiences a force (Figure 9.1). The magnitude of the force (F) is proportional to the strength of magnetic field (B), length of the conductor (L), and current (i), $F = B \cdot i \cdot L$. The direction of force can be found by Flemming's left hand law (Figure 9.1). Using a coiled conductor (Figure 9.2) leads to a torque, with direction dependent upon the direction of current (and voltage). This is the physical basis for torque production in a motor.

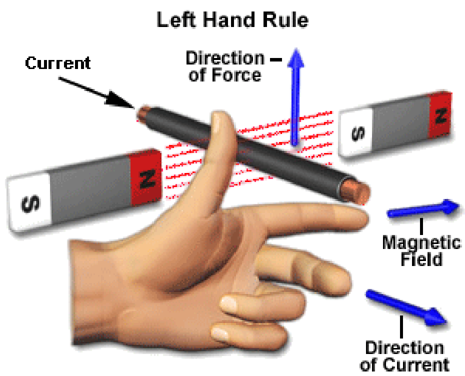


Figure 9.1: Fleming's left hand rule

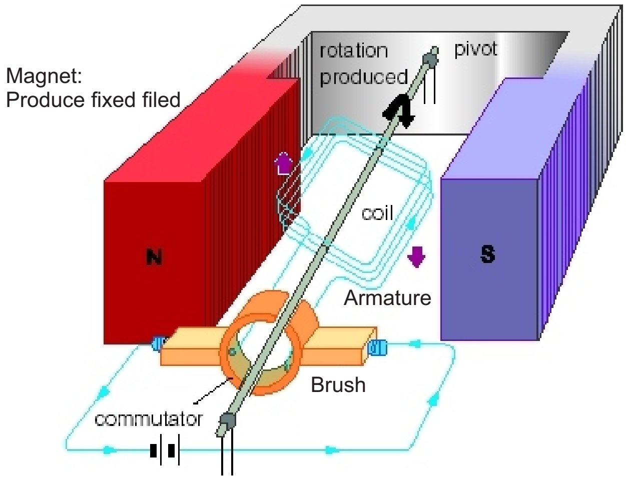
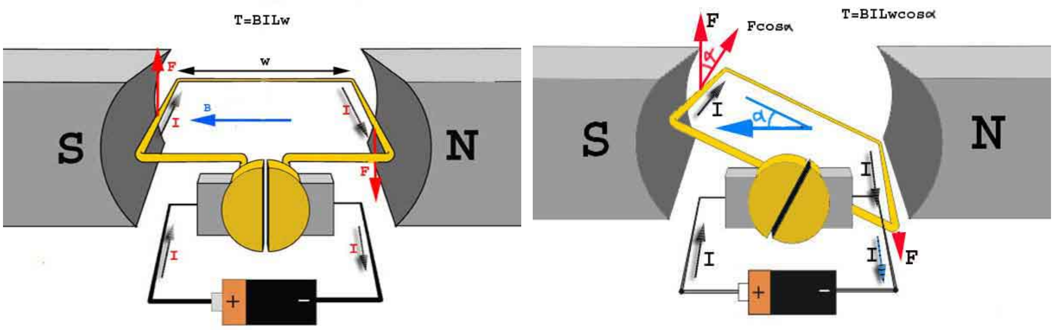


Figure 9.2: Simple DC Motor Schematics

Figure 9.2 depicts a simplified version of an electric motor with:
Fixed field: Magnetic field, produced by magnets.
Armature: Coil of wire used to produce torque. When current flows through the coil, the armature conductor experiences a torque.
Commutation: The torque produced by the coil and magnet is dependent on their relative orientation. When the coil above is horizontal, torque is maximized. When it is vertical, torque is zero. To keep a steady torque, DC motors have many coils and many magnets, each with slightly different orientation. As the rotor moves, currents are switched on and off, in each direction, by the commutator. In DC motors, it is often composed of split rings and contacted with a *brush*.
Stator: Stationary part of the motor. Often permanent magnets in DC motors.
Rotor: Rotating part of the motor. Often conductors in DC motors.

Figure 9.3 illustrates the operating principle in more detail. When an armature is placed in a fixed field and current flows through the coil, the left side of the coil experiences upward force and right side of the coil experiences downward force (Figure 9.3a). As the rotor moves, the tangent component of the force, responsible for torque generation, becomes $F_{tan} = B \cdot i \cdot L \cos\alpha$, where α is the angle of the rotor with respect to the ideal orientation (Fig 9.3b). When α is 90 degrees, the force is zero (although inertia may carry the rotor through this point). Once the armature has rotated more than 90 degrees, the commutator switches the direction of current such that the direction of the force on each side of the coil remains the same and the rotor experiences the same direction of torque. A motor with multiple coils, each with a different orientation, can generate approximately constant torque (Figure 9.4), $\tau_{em} \approx K_a \cdot i$, where K_a is the *torque constant*.



(a) Simple DC motor with $\alpha = 0$ (b) Simple DC motor with $\alpha \neq 0$

Figure 9.3: Simple DC motor operation explanation

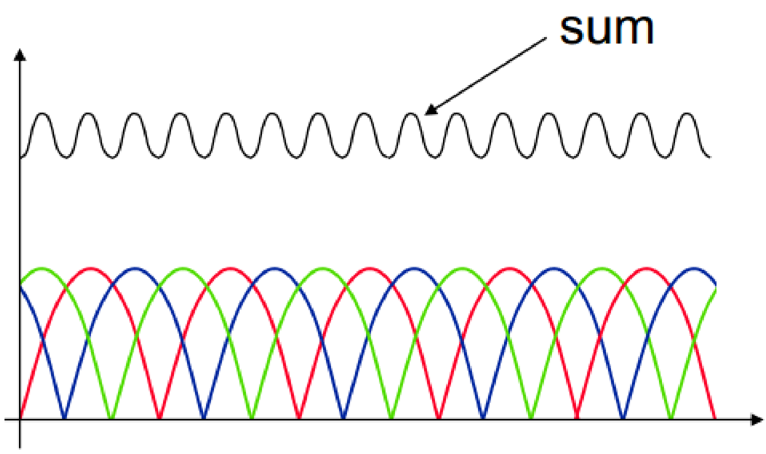


Figure 9.4: Multiple coil torque superposition

When an armature conductor moves with velocity orthogonal to a magnetic field, a voltage is generated. This is called *back EMF*. The magnitude of the voltage (V_b) is proportional to the strength of the magnetic field (B), length of the conductor (L), and the velocity of the conductor normal to the magnetic field (v), or $V_b = B \cdot L \cdot v$. The direction of the induced voltage can be found using Flemming's right hand law. It will fight the applied voltage if the motor is rotating in the same direction as the induced torque. In a DC motor with commutation and many coils and magnets, the back EMF is approximately proportional to angular velocity ($\dot{\theta}_m$), or $V_b \approx K_b \cdot \dot{\theta}_m$, where $K_b \approx K_a$ is the *velocity constant*.

In AC motors, the input AC current is usually applied to the wire coils on the stator to provide a varying magnetic field as shown in Figure 9.5. The rotor here

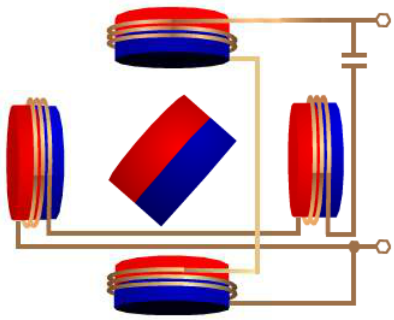


Figure 9.5: Simple AC motor

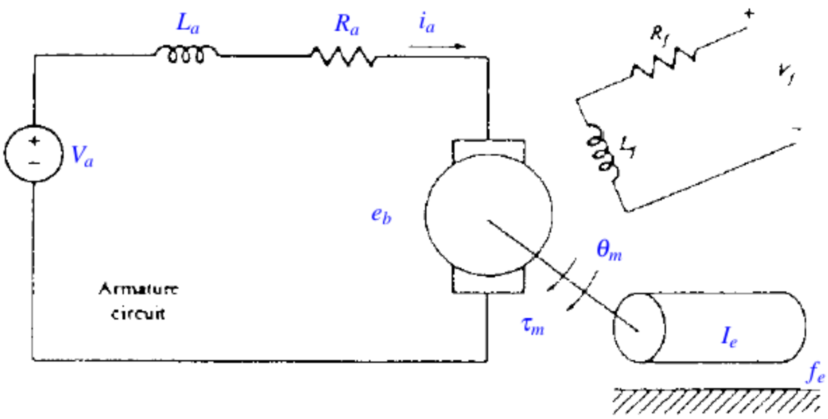


Figure 9.6: DC motor circuit

is usually a permanent magnet. The mechanism of torque production is magnetic force. Note that due to the alternate current directions, there are alternate magnetic field directions, so no brush is needed. Constant torque can be produced by carefully designing the phase differences between different coil pairs.

AC motors are commonly used for high power applications while DC motors are popular in robotics applications due to ease in control.

9.2 DC Motor Modeling

A circuit diagram for an armature-controlled DC motor is shown in Figure 9.6. The motor shaft is coupled to a load through the gear train as illustrated in Figure 9.7. The motor parameters are:

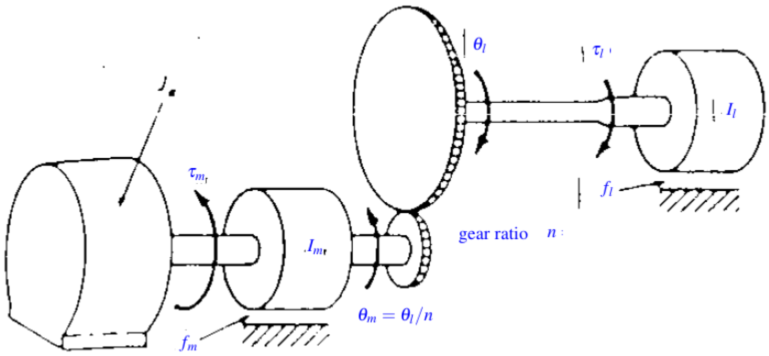


Figure 9.7: DC Motor Gear Train

- V_a is the applied armature voltage,
- i_a is the current through the motor coil in the armature,
- K_a is the motor-torque constant,
- $\tau_{em} = K_a \cdot i_a$ is the electromechanical torque due to current in the motor
- θ_m is the angular displacement of the motor rotor,
- K_b is the back EMF constant (equal to K_a in an ideal motor),
- $V_b = K_b \cdot \dot{\theta}_m$ is the back emf,
- R_a is the armature resistance,
- L_a is the armature inductance,
- R_G is the gear ratio,
- $\theta_o = {}^1\!/_{R_G} \cdot \theta_m$ is the angular displacement of the gearbox output shaft,
- J_m is the rotational mass moment of inertia of the motor,
- J_L is the rotational mass moment of inertia of the load,
- $J_e = J_L + R_G^2 \cdot J_m$ is the effective moment of the inertia of the motor and load referred to the output shaft,
- b_m is the viscous friction coef. of the motor,
- b_o is the viscous friction coef. of the output shaft,
- $b_e = b_o + R_G^2 \cdot b_m$ is the effective viscous friction coefficient of the combined motor and load referred to the output shaft.
- τ_o output torque at the load side

The electronic portion of the system can be modeled using Kirchoff's voltage law:

$$\sum V = V_a(t) - V_b(t) = R_a \cdot i_a(t) + L_a \frac{di_a(t)}{dt}$$

or, including the definition of back EMF,

$$V_a(t) - K_b \dot{\theta}_m(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt}$$

For purposes of dynamical modeling of the system, we wish to obtain equations of motion in which the highest order derivative is isolated on one side. Here, this will be the rate of change in current:

$$\frac{d}{dt} i_a(t) = \frac{1}{L_a} \cdot \left(V_a(t) - K_b \dot{\theta}_m(t) - R_a i_a(t)\right) \tag{9.1}$$

The mechanical portion of the system can be modeled using Newton's second law. This relationship must be consistently calculated on one side of the gearbox to avoid mismatch between torque and inertia. On the output shaft side, the torques include electromechanical, damping, and external output contributions:

$$\sum \tau = R_G \cdot \tau_{em} - \tau_o - b_e \dot{\theta}_o = J_e \ddot{\theta}_o$$

or, including the definition of the electromechanical torque,

$$R_G K_a \cdot i_a(t) - \tau_o - b_e \dot{\theta}_o = J_e \ddot{\theta}_o$$

Isolating the highest derivative, here acceleration of the output shaft, we have:

$$\ddot{\theta}_o = \frac{1}{J_e} \cdot \left(R_G K_a \cdot i_a(t) - \tau_o - b_e \dot{\theta}_o\right) \tag{9.2}$$

Equations 9.1 and 9.2 are a set of coupled equations of motion that fully describe the dynamics of the system. The terms in these equations break down as:

- **State variables** are current, i_a, output angle, θ_o, and output velocity, $\dot{\theta}_o$.
- **System inputs** are applied voltage, V_a, and output shaft torque, τ_o.
- **Constant parameters** include coil inductance, L_a, motor constants, K_a and K_b, coil resistance, R_a, gear ratio, R_G, rotor and load inertias, J_e (or J_m and J_L), and motor and load damping, b_e (or b_m and b_o).

A state-space representation of this system can be defined with state x, where:

$$
\begin{aligned}
x_1 &= \theta_o \\
x_2 &= \dot{\theta}_o \\
x_3 &= i_a
\end{aligned}
$$

and input u, where:

$$
\begin{aligned}
u_1 &= \tau_o \\
u_2 &= V_a
\end{aligned}
$$

Equations 9.1 (noting $\dot{\theta}_m = R_G \cdot \dot{\theta}_o$) and 9.2 can therefore be written as:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{b_e}{J_e} x_2 + \frac{R_G K_a}{J_e} x_3 - \frac{1}{J_e} u_1 \\
\dot{x}_3 &= -\frac{K_b \cdot R_G}{L_a} x_2 - \frac{R_a}{L_a} x_3 + \frac{1}{L_a} u_2
\end{aligned}
$$

or, in typical state-space form, as

$$\dot{x} = Ax + Bu$$

where

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b_e}{J_e} & \frac{R_G K_a}{J_e} \\ 0 & -\frac{K_b \cdot R_G}{L_a} & -\frac{R_a}{L_a} \end{bmatrix}
$$

$$
B = \begin{bmatrix} 0 & 0 \\ -\frac{1}{J_e} & 0 \\ 0 & \frac{1}{L_a} \end{bmatrix}
$$

This organization is well-suited to numerical simulation using ode45 in Matlab.

9.3 Motor Selection

The most important criteria in motor selection are often related to maximum torque, velocity and power. For motor selection, we can usually use a simplified version of the model presented in the previous section.

Inductance is often neglected in such a simplification, since it is usually very small (see the red box in Figure 9.8). Notice that without inductance, the electronics equation is no longer dynamic. In other words, as soon as a voltage is applied the current will instantly change, with a value completely prescribed by applied voltage, rotational speed, and constant parameters. In the real system, current dynamics are typically very fast compared to the mechanical dynamics, making this a strong approximation. Notice that this will reduce the size of our state space from three (θ_o, $\dot{\theta}_o$ and i_a) to two (θ_o and $\dot{\theta}_o$). It is often convenient to then consider current as an input to the system, rather than voltage. The governing equation for the electrical portion of the system becomes:

$$V_a = R_a \cdot i_a + R_G K_b \cdot \dot{\theta}_o$$

It is also convenient to consider the system at steady state, i.e. $\ddot{\theta}_o \approx 0$, and to neglect friction, i.e. $b_e \approx 0$. This approximation is weaker than that for inductance, and should be used carefully. The governing equation for the mechanical side of the system then becomes:

$$R_G \tau_{em} = \tau_o$$

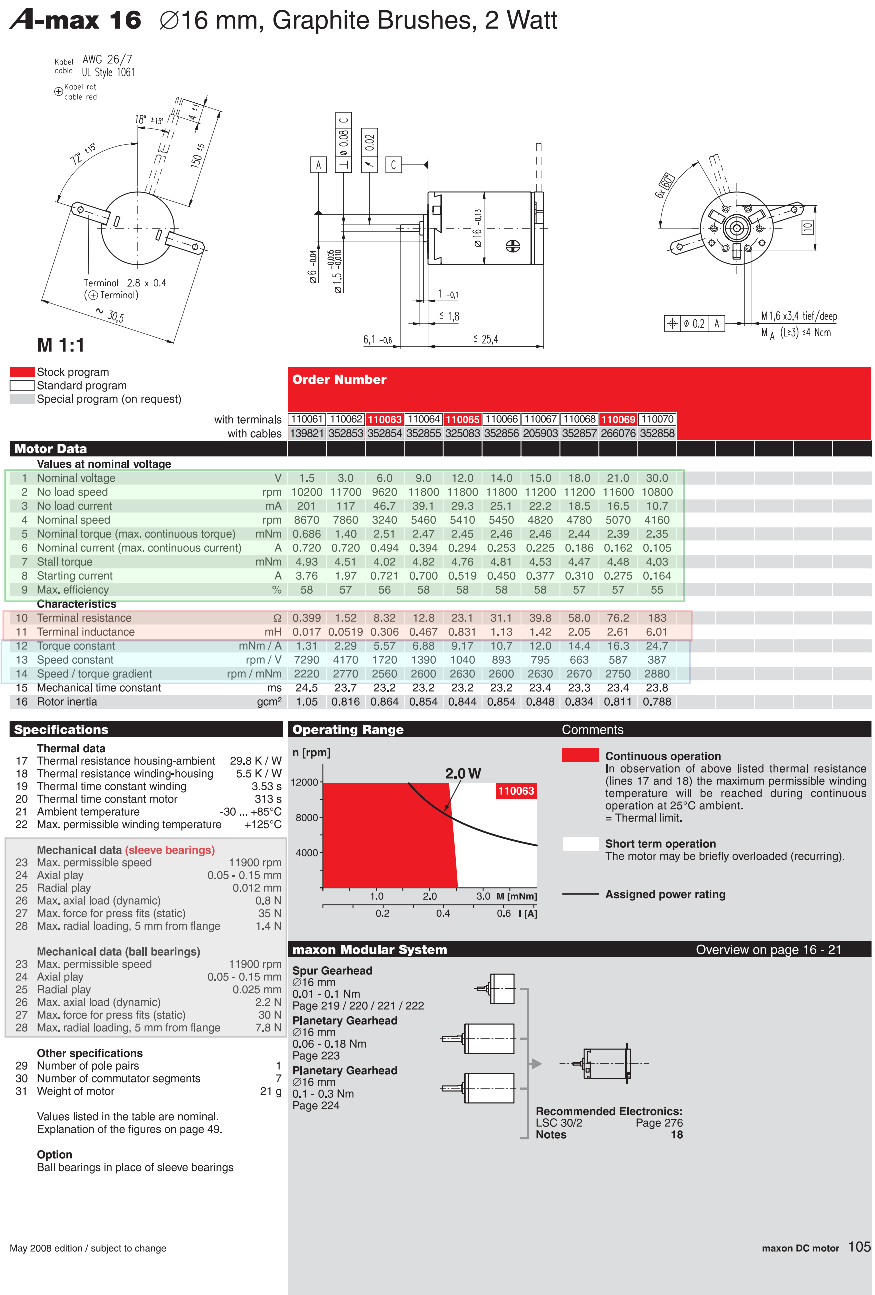
or, substituting for τ_{em} and solving for i_a,

$$i_a = \frac{\tau_o}{R_G K_a}$$

This leads to a simplified, steady-state equation relating applied voltage, output torque and output speed:

$$\tau_o = \frac{R_G K_a}{R_a} \cdot V_a - \frac{R_G^2 K_a K_b}{R_a} \cdot \dot{\theta}_o \tag{9.3}$$

That is, for a given applied voltage, there is a linear relationship between output torque and motor speed at steady state. Using this relationship and the constants for a particular motor (e.g. from the pink box in the datasheet in Figure 9.8) we can create torque-speed curves as in Figure 9.9. We can also solve for key outcomes under simplified circumstances, including peak torque, peak speed and peak power, which is useful for motor selection.



A-max 16 ∅16 mm, Graphite Brushes, 2 Watt

maxon A-max

M 1:1

Stock program
Standard program
Special program (on request)

Order Number

	with terminals	110061	110062	110063	110064	110065	110066	110067	110068	110069	110070
	with cables	139821	352853	352854	352855	325083	352856	205903	352857	266076	352858
Motor Data											
Values at nominal voltage											
1 Nominal voltage	V	1.5	3.0	6.0	9.0	12.0	14.0	15.0	18.0	21.0	30.0
2 No load speed	rpm	10200	11700	9620	11800	11800	11800	11200	11200	11600	10800
3 No load current	mA	201	117	46.7	39.1	29.3	25.1	22.2	18.5	16.5	10.7
4 Nominal speed	rpm	8670	7860	3240	5460	5410	5450	4820	4780	5070	4160
5 Nominal torque (max. continuous torque)	mNm	0.686	1.40	2.51	2.47	2.45	2.46	2.46	2.44	2.39	2.35
6 Nominal current (max. continuous current)	A	0.720	0.720	0.494	0.394	0.294	0.253	0.225	0.186	0.162	0.105
7 Stall torque	mNm	4.93	4.51	4.02	4.82	4.76	4.81	4.53	4.47	4.48	4.03
8 Starting current	A	3.76	1.97	0.721	0.700	0.519	0.450	0.377	0.310	0.275	0.164
9 Max. efficiency	%	58	57	56	58	58	58	58	57	57	55
Characteristics											
10 Terminal resistance	Ω	0.399	1.52	8.32	12.8	23.1	31.1	39.8	58.0	76.2	183
11 Terminal inductance	mH	0.017	0.0519	0.306	0.467	0.831	1.13	1.42	2.05	2.61	6.01
12 Torque constant	mNm / A	1.31	2.29	5.57	6.88	9.17	10.7	12.0	14.4	16.3	24.7
13 Speed constant	rpm / V	7290	4170	1720	1390	1040	893	795	663	587	387
14 Speed / torque gradient	rpm / mNm	2220	2770	2560	2600	2630	2600	2630	2670	2750	2880
15 Mechanical time constant	ms	24.5	23.7	23.2	23.2	23.2	23.2	23.4	23.3	23.4	23.8
16 Rotor inertia	gcm^2	1.05	0.816	0.864	0.854	0.844	0.854	0.848	0.834	0.811	0.788

Specifications

Thermal data
17 Thermal resistance housing-ambient 29.8 K / W
18 Thermal resistance winding-housing 5.5 K / W
19 Thermal time constant winding 3.53 s
20 Thermal time constant motor 313 s
21 Ambient temperature -30 ... +85°C
22 Max. permissible winding temperature +125°C

Mechanical data (sleeve bearings)
23 Max. permissible speed 11900 rpm
24 Axial play 0.05 - 0.15 mm
25 Radial play 0.012 mm
26 Max. axial load (dynamic) 0.8 N
27 Max. force for press fits (static) 35 N
28 Max. radial loading, 5 mm from flange 1.4 N

Mechanical data (ball bearings)
23 Max. permissible speed 11900 rpm
24 Axial play 0.05 - 0.15 mm
25 Radial play 0.025 mm
26 Max. axial load (dynamic) 2.2 N
27 Max. force for press fits (static) 30 N
28 Max. radial loading, 5 mm from flange 7.8 N

Other specifications
29 Number of pole pairs 1
30 Number of commutator segments 7
31 Weight of motor 21 g

Values listed in the table are nominal.
Explanation of the figures on page 49.

Option
Ball bearings in place of sleeve bearings

Operating Range

Comments

Continuous operation
In observation of above listed thermal resistance (lines 17 and 18) the maximum permissible winding temperature will be reached during continuous operation at 25°C ambient.
= Thermal limit.

Short term operation
The motor may be briefly overloaded (recurring).

Assigned power rating

maxon Modular System Overview on page 16 - 21

Spur Gearhead
∅16 mm
0.01 - 0.1 Nm
Page 219 / 220 / 221 / 222

Planetary Gearhead
∅16 mm
0.06 - 0.18 Nm
Page 223

Planetary Gearhead
∅16 mm
0.1 - 0.3 Nm
Page 224

Recommended Electronics:
LSC 30/2 Page 276
Notes **18**

May 2008 edition / subject to change

maxon DC motor 105

Figure 9.8: Maxon motor datasheet

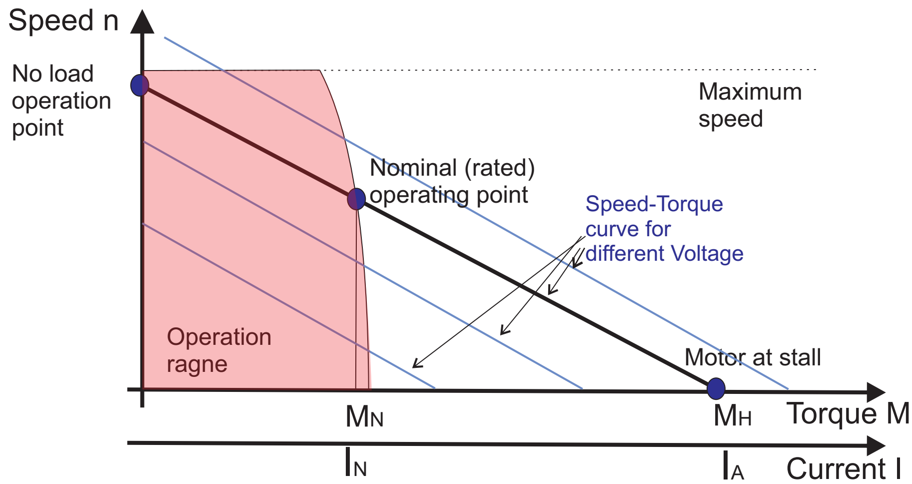


Figure 9.9: Speed torque curve

- **Peak torque:** From equation 9.3, we can see that output torque will increase as applied voltage (and therefore coil current) is increased, and as output velocity is decreased. For $\dot{\theta} = 0$, we have:

$$i_{a_{max}} \approx V_a/R_a \rightarrow \textit{maximum (stall) current}$$
$$\tau_{o_{max}} \approx R_G K_a \cdot i_{a_{max}} = \frac{R_G K_a V_a}{R_a} = \tau_{stall} \rightarrow \textit{maximum (stall) torque}$$

  In real motors, we cannot apply arbitrarily large voltage or current indefinitely without causing damage. With high currents, the motor will experience Joule heating in the coil at a rate of $P = i_a \cdot V_a = i_a^2 \cdot R_a$. As the temperature of the motor increases, the permanent magnets will eventually lose their magnetic effect, or *demagnetize*. Manufacturers typically list the maximum current that can be constantly applied without overheating the motor as the nominal torque, τ_{nom}. Higher current can, however, be applied for short durations, provided that there is enought time between bursts for the motor to cool down.

  Keep in mind that this approximation does not account for friction in the gearbox (often listed in terms of efficiency) or acceleration of motor and gearbox elements.

- **Maximum speed:** From equation 9.3, we can see that motor velocity will increase as applied voltage is increased and output torque is decreased. In the idealized case that $\tau_o = 0$, we have:

$$\dot{\theta}_{o_{max}} \approx \frac{1}{R_G \cdot K_b} \cdot V_a \rightarrow \textit{maximum (no load) speed}$$

In real motors, we cannot apply arbitrarily large voltage or allow arbitrarily high speeds without causing damage. At high voltage, the brushes in brushed DC motors will be slightly damaged with each switch of the commutator, reducing motor life. At very high voltage, the coil insulator can break down, even in brushless motors. At high speeds bearings may be damaged, resulting in higher friction and shorter life. Manufacturers will typically specify a maximum applied voltage, or nominal voltage V_{nom}, and maximum speed, or ω_{max}, that can be continually applied without reducing motor life. In some applications, it is worthwhile to sacrifice motor endurance for higher speed (and power) in a smaller, lighter motor.

Keep in mind that this approximation does not account for friction, which results in the need for finite current even with no external load, thereby reducing peak speed.

- **Maximum power:** The optimal power output occurs with a combination of medium torque and speed. This is not simply the product of maximum torque and maximum speed, since these two events do not occur under the same conditions. Manufacturers typically list (see, for example, the green box in Figure 9.8) the continual torque, τ_{nom}, and velocity, $\dot{\theta}_{nom}$, that correspond to the maximum continual power:

$$P_{max} \approx \tau_{nom} \cdot \dot{\theta}_{nom} \rightarrow \text{maximum continuous power}$$

In real motors, the same practical limits on applied voltage and coil current apply as in the preceeding discussions of peak motor speed and torque. It is therefore possible to achieve higher average power than the rated nominal power by increasing applied voltage and operating speed, but sacrificing motor life. It is also possible to achieve short bursts of very high power without sacrificing motor life, provided that the operating speed and voltage are kept below manufacturer limits and the current exceeds the nominal rating only for short bursts. Finally, one can achieve very high power output for short periods of time by exceeding the voltage, speed and current limits simultaneously, at a cost of reduced motor life.

Keep in mind that the above simplification does not take gearbox losses into consideration.

- **Efficiency:** Efficiency, η, is the ratio of power output to power input, or $\eta = P_{output}/P_{input}$. For DC motors, η is typically around 0.8. In other words, for each Joule of electrical energy supplied, under optimal conditions, 0.8 Joules of mechanical work are performed. Gearboxes have additional losses, with efficiencies from 0.4 to 0.8. In other words, for each Joule of mechanical work done on the gearbox by the motor, 0.4 to 0.8 Joules of work are done by the gearbox on whatever is attached to the output shaft.

Motor inefficiency is captured by the mismatch between K_a and K_b, and requires no further modeling. Gearbox losses, however, require a little more consideration. Lost power occurs as lost torque between the input and output of the gearbox, due to friction. (Power losses cannot come from lost velocity, of course, since the elements are constrained to move together). If we approximate the effect as Coloumb friction, resistance will scale with normal forces between teeth, which in turn will scale with transmission torque. In other words, the transmitted toque will be some fraction of the ideal (frictionless) transmitted torque, and will scale linearly. Since power losses occur due to torque loss, the linear coefficient is therefore approximately equal to η. Returning to equation 9.2, we can adjust the sum of torques to account for this loss:

$$\sum \tau = R_G \cdot \eta \cdot \tau_{em} - \tau_o - b_e \dot{\theta}_o = J_e \ddot{\theta}_o$$

or, in terms of our preferred state variables,

$$\ddot{\theta}_o = \frac{1}{J_e} \cdot (R_G K_a \cdot \eta \cdot i_a(t) - \tau_o - b_e \dot{\theta}_o) \tag{9.4}$$

- **Dynamic response:** The complete dynamic behavior of the system, including transient responses to changes in applied voltage or external loads, requires analysis using the complete equations of motion, i.e. 9.1 and 9.2. The *time constant* of the motor listed by the manufacturer characterizes the time it takes for the rotor to accelerate towards the no-load speed when a step increase in voltage is applied. This only applies to the motor; inertia from a gearbox or external inertia will slow things down. The *bandwidth* of a gearmotor system characterizes the maximum frequency for which the system is responsive to changes in, e.g., desired motor speed. In most applications using electric motors, it is useful in the detailed design phase to perform a full simulation of motor behavior under expected operating conditions.

- **Other mechanical considerations:** Most motors and gearboxes are not designed to sustain large radial loads on their output shafts. During motor integration, be sure to check the maximum radial and axial loads recommended by the manufacturer (such as those in Figure 9.8.

9.4 Acknowledgments

Thanks to Juanjuan Zhang, Myunghee Kim, Kirby Witte and Evan Dvorak for contributions to this chapter.