Engineering Design I: Methods and Skills

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Chapter 3

Stress Analysis for Design

As a student in Mechanical Engineering or a similar field, you will have taken a course or courses covering the fundamentals of stress and strength of materials. In this chapter, we will apply these concepts in the context of the analysis of a candidate component. Although this chapter reviews some concepts of stress and strength, uninitiated readers are encouraged to first read a book dedicated to the subject, such as *Mechanics of Materials* by Steif [2012]. Such texts provide a wider set of analytical tools for mechanical design problems, and provide the technical foundation for developing intuition for component strength.

3.1 Simplified Stress Analysis

We will use the term "simplified stress analysis" to refer to analysis using simple, established analytical models. This approach quickly develops fundamental relationships between peak stresses and design parameters using hand analysis of (drastically) simplified models. These relationships can be inverted to find analytically optimal design parameters, providing a strong foundation for detailed design. It is most commonly used early in the design process and at the initial phases of re-design iterations.

3.2 Detailed Stress Analysis

We will use the term "detailed stress analysis" to refer to analysis using detailed computational models. This approach addresses nearly the full complexity of a design, typically using Finite Element Analysis (FEA) software tools, allowing for iterative refinement of design details. It is most commonly used later in the design process, after simple analysis and before prototyping.

3.3 Example: This Old I-Beam

In this example, we will perform simple and detailed stress analysis on an I-beam (Figure 1) and compare the answers. An accompanying SolidWorks part file can be found on Blackboard. In this exercise we will make the following assumptions and definitions:

- Cantilever loading: rigidly supported on one end, load = F at the other
- I-beam length = L, height = h, base width = b, and thickness = t
- Holes along center of the web have diameter = D
- Holes along the flanges have diameter = d
- Material: Alloy Steel ASTM A36, a ductile material

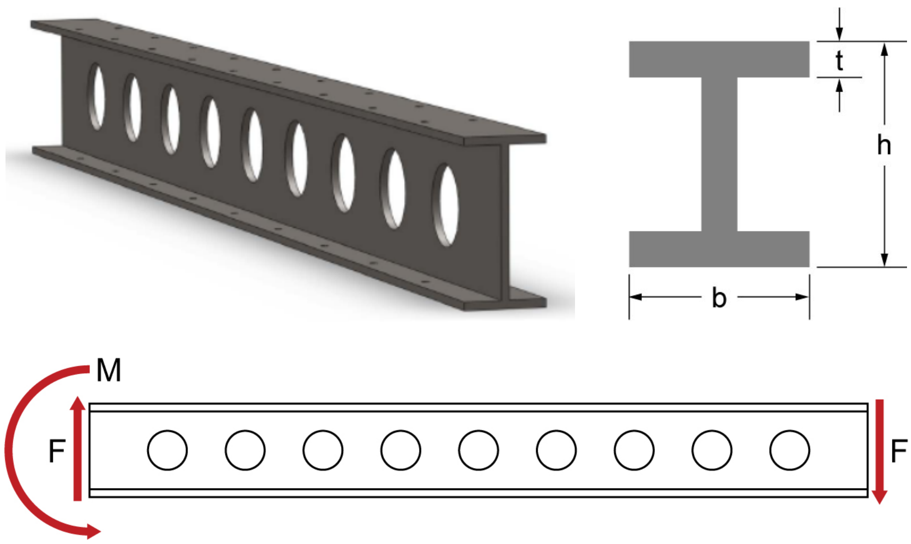


Figure 1. *Left:* image of the I-beam. *Right:* diagram of the I-beam's cross-section. *Bottom:* free body diagram of the beam in cantilever loading.

3.3.1 Simple Stress Analysis

Intuition check: Where will the stresses be highest in this I-beam? Chew this over for a moment before continuing.

Models to be used: Based on our familiarity with the cantilevered beam problem, we see quickly that the Euler-Bernoulli approximation ($\sigma = M \cdot y/I$) will be helpful here. This model suggests that stresses will be highest near the cantilever point, at the upper and lower edges of the I-beam. We also recognize that the holes will cause some form of stress concentration, for which we can look up a concentration factor. This suggests the highest stress will be at the edge of a hole on the flange near the fixed end.

Analysis objective: This is the type of problem you've encountered many times as an engineering student. For known geometric parameters (h, b, t, D), loading (F), and material properties (σ_y), what is the Factor of Safety (F.O.S.)?

Design objective: This is a new problem, specific to design. What *should* the parameters be? For example, if we assume material properties, loading, F.O.S., and some geometric parameters, what should the other geometric parameters be? Which parameters have the most influence over outcomes, such as beam mass? We will see how simple analysis results can be inverted to answer these questions.

Step 1: Approximate bending stress

Let's use a simple model of stress in a beam loaded in bending to get a first order approximation of the peak stress in our I-beam. The peak stress can be estimated using Euler–Bernoulli beam theory [e.g. Budynas and Nisbett, 2006]:

$$\sigma_m \approx \frac{M_{max} \cdot y_{max}}{I} \tag{3.1}$$

Load analysis with a Free Body Diagram would show that the maximum bending moment is $M_{max} = F \cdot L$ on the fixed end. The maximum distance from the centroid is $y_{max} = h/2$. I is the area moment of inertia of the beam. Unfortunately, we don't know a simple formula for this shape, so we will have to come up with a decent approximation ourselves.

Step 1.b: Area moment of inertia of the I-beam

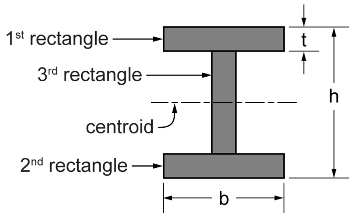
Let's try to derive I for our I-beam. The second moment of area of a rectangular region about its neutral axis is [e.g. Budynas and Nisbett, 2006]:

$$I_{rect} = \frac{1}{12} w \cdot h^3$$

where w and h are the width and height of the rectangle respectively. Of course, the I-beam cross section includes at least three rectangles, two of which do not share a centroid with the beam as a whole. The *parallel axis theorem* [e.g. Beer et al., 2009], tells us that the second moment of area of a shape about an axis parallel to its neutral axis is:

$$I = I_{neutral} + A \cdot d^2,$$

where $I_{neutral}$ is the area moment of inertia about the shape's neutral axis, A is the area of the shape and d is the distance between the shape's neutral axis and the desired axis (here the centroid of the beam).



Treating each rectangle that comprises the cross-section separately, the combined I-beam area moment of inertial is therefore:

$$I_{total} = 2 \cdot \left[\frac{1}{12} b \cdot t^3 + \left(\frac{h}{2} - \frac{t}{2} \right)^2 \cdot b \cdot t \right] + \frac{1}{12} t (h - 2t)^3$$

where the first term captures the flanges and the second accounts for the web.

While strictly correct, this solution is complex. That makes it hard to interpret and difficult to manipulate or invert. Let's ***severely*** reduce the problem until we get a very simple equation that captures the most fundamental relationships and is accessible to further hand analysis. We will use the assumptions that $t \ll b$, $t \ll h$, that b is of similar magnitude to h, and even neglect terms with smaller constants:

$$\begin{aligned}
I_{total} &= 2 \cdot \left[\frac{1}{12} b \cdot t^3 + \left(\frac{h}{2} - \cancel{\frac{t}{2}} \right)^2 \cdot b \cdot t \right] + \frac{1}{12} t (h - \cancel{2t})^3 \\
&\qquad \text{(on the basis that } h \text{ will dominate these terms)} \\
&\approx 2 \cdot \left[\cancel{\frac{1}{12} b \cdot t^3} + \left(\frac{h}{2} \right)^2 \cdot b \cdot t \right] + \frac{1}{12} t (h)^3 \\
&\qquad \text{(since } t^3 \text{ will be small)} \\
&\approx \frac{1}{2} t b h^2 + \cancel{\frac{1}{12} t h^3} \\
&\qquad \text{(because the coefficient is smaller)} \\
&\approx \frac{1}{2} b t h^2
\end{aligned} \tag{3.2}$$

This is now a simple approximation of I suitable for further hand analysis. Let's substitute Equation (3.2) back into Equation (3.1) to obtain a complete symbolic relationship in terms of our design parameters:

$$\sigma_{max} \approx \frac{F \cdot L \cdot \frac{1}{2}h}{\frac{1}{2}bth^2} = \frac{F \cdot L}{b \cdot t \cdot h}. \tag{3.3}$$

Step 2: Factor of Safety

Recall that the factor of safety is the ratio of the capacity of the system (*numerator*) to the expected operating conditions (*denominator*), in this case the ratio of the failure stress to the expected maximum stress. We do not wish our beam to plastically deform, therefore exceeding the yield stress, σ_y, would not be acceptable. In other words:

$$F.O.S. = \frac{\sigma_y}{\sigma_{max}}$$

Combining with Equation (3.2) we have:

$$F.O.S. = \frac{\sigma_y \cdot b \cdot t \cdot h}{F \cdot L} \tag{3.4}$$

Reflect and interpret: The above equation tells us the relationship between a critical outcome, $F.O.S.$, the material chosen, σ_y, the applied load, F, and the most basic geometric parameters of our design. We can see that increasing b, t, or h would all equally increase the factor of safety, which might contradict your intuition prior to this analysis (e.g. that h would dominate). In keeping with our intuition, we also see that minimizing L will maximize the safety factor. But what if we have a target factor of safety in mind? We will return to this question momentarily...

Recriminations: What about shear stress?

So far, we have plunged forward on the assumption that bending stresses dominate. But isn't the beam experiencing shear loading as well? Let's calculate peak shear and compare to the stresses estimated by our bending model. The average shear stress can be expressed simply as

$$\tau_{avg} = \frac{F}{A}$$

If we calculate the combined cross-sectional area and simplify, we have

$$\begin{aligned} A &= 2 \cdot b \cdot t + t \cdot (h - 2t) \\ &\approx 2bt + ht - \cancel{2t^2}. \end{aligned}$$

Thus,

$$\tau_{avg} \approx \frac{F}{2b \cdot t + h \cdot t}$$

A slightly more accurate peak shear stress in an I-beam can be calculated based on the cross-sectional area of the web alone [Budynas and Nisbett, 2006]

$$\tau_{max} \approx \frac{F}{A_{web}} \approx \frac{F}{h \cdot t} \tag{3.5}$$

We can see that τ_{avg} is lower than τ_{max} (there is an additional term of $2bt$ in the denominator), making τ_{max} a more conservative estimate.

But how does the peak shear stress compare to the peak bending stress? If we divide Equation (3.3) by Equation (3.5), and set aside differences between shear and axial stress, we obtain a ratio of bending to shear stress in the beam of

$$\frac{\sigma_{max}}{\tau_{max}} \approx \frac{\frac{F \cdot L}{b \cdot t \cdot h}}{\frac{F}{t \cdot h}} = \frac{L}{b}$$

Reflect and interpret: Since we expect our beam to be longer than it is wide, the above relationship tells us to expect the axial stresses induced by bending to be larger than those due to shear loading. Interestingly, because the I-beam is well-designed to withstand bending (i.e. to have a large area moment of inertia for a small amount of material), shear actually plays a bigger role than it would in a beam with rectangular cross section, where the ratio would work out to $6 \cdot L \cdot h^{-1}$.

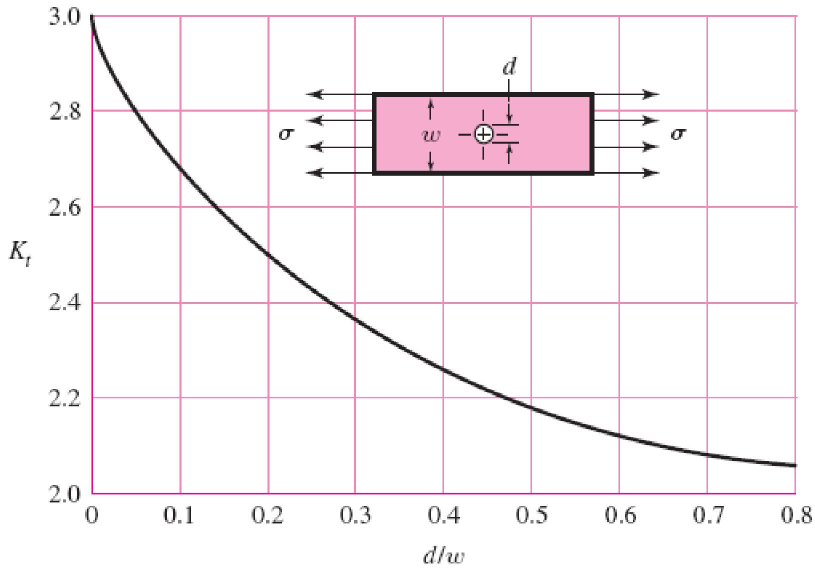
In other words, as you optimize for one failure mode, perhaps counter-intuitively, secondary failure modes become more relevant.

Part 3: Stress concentrations from holes in the beam

Does it matter that our old I-beam has a bunch of holes in it? As a first approximation (Steps 1 & 2), no. However, we can increase the accuracy of our stress estimate, with only a minor increase in model complexity, by considering these holes as stress concentrators. We recall from our stress course that a theoretical, or geometric, stress-concentration factor, K_t [e.g. Dieter and Schmidt, 2009] relates the maximum stress at a discontinuity to the nominal stress:

$$K_t = \frac{\sigma_{max}}{\sigma_{nom}}.$$

Checking our texts [e.g. Beer et al., 2009, Budynas and Nisbett, 2006, Steif, 2012] for a relevant look-up chart, we find the following:



Judging from Figure 1, we guess that for the holes of interest (those along the top and bottom flanges) the key ratio is $d/b \approx 0.1$. This gives us $K_t \approx 2.7$, no small multiplier.

Part 4: Substitutions

Notice that we have learned quite a lot without substituting numerical values into any of our equations! Of course, to complete the analysis portion of our exercise, we need to calculate factors of safety. Let's use the following reasonable values to generate numerical answers. Let's assume the beam is constructed of ASTM A36 alloy steel, a ductile material with $sigma_y \approx 36,000$ psi; a load $F = 5,000$ lbf was applied; and the beam has $L = 100$ in., $h = 10$ in., $b = 5$ in., $t = 0.5$ in., $D = 5$ in., and $d = 0.5$ in. Using the bending model and neglecting stress concentrations, we have:

$$\sigma_m \approx 20,000 \frac{lb}{in^2} = 20\, ksi,$$

thus the Factor of Safety based on the maximum bending stress is

$$F.O.S. = \frac{\sigma_y}{\sigma_m} \approx \frac{36\,ksi}{20\,ksi} \approx 1.8$$

If we include the stress concentration factor, we have

$$F.O.S.^* \approx \frac{\sigma_y}{K_t \cdot \sigma_{nom}} = \frac{36}{2.7 \cdot 20} = \frac{36}{54} \approx 0.7.$$

Reflect and interpret: Once provided with numerical values, we quickly determined a factor of safety. However, these numbers do not contain the rich relationships available in our symbolic equations and do not allow inverse analysis for design, so it's a good thing we didn't substitute too early. Between the two factors of safety we did calculate, which is more accurate or useful? There *will* be stress concentrations due to the holes on the flanges, and so including K_t results in a more accurate estimate of peak stress. However, in the case of a ductile material undergoing static (non-cyclic) loading, we can most likely neglect the stress concentrators when calculating factor of safety, since strain hardening and local plastic deformations will both act against failure at the concentrator.

3.3.2 Detailed stress analysis

These same analyses can be performed on a detailed computational model of the I-beam, for example using the Simulation feature in SolidWorks. Accompanying this document, you will find a SolidWorks part model that includes such an analysis. Below are screen shots of the results of this analysis.

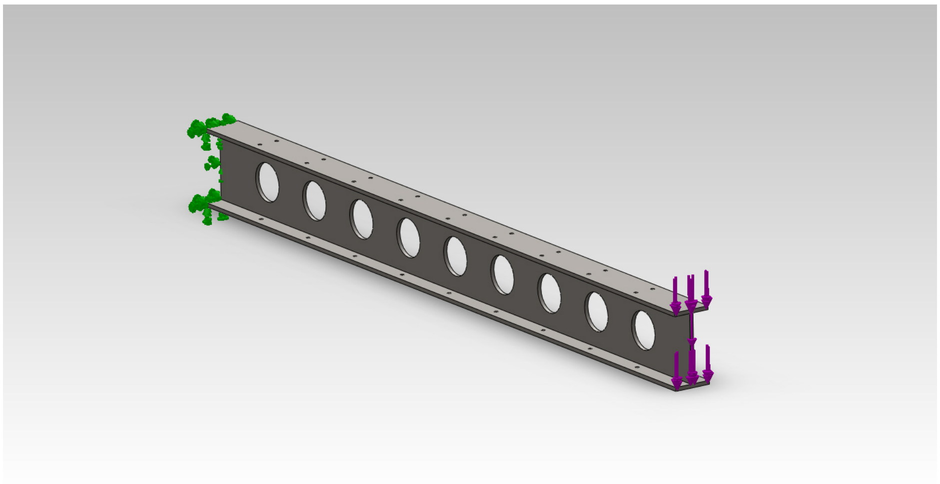


Figure 2. Solid model of the I-beam with fixtures (green arrows at left) and loading (pink arrows at right) shown.

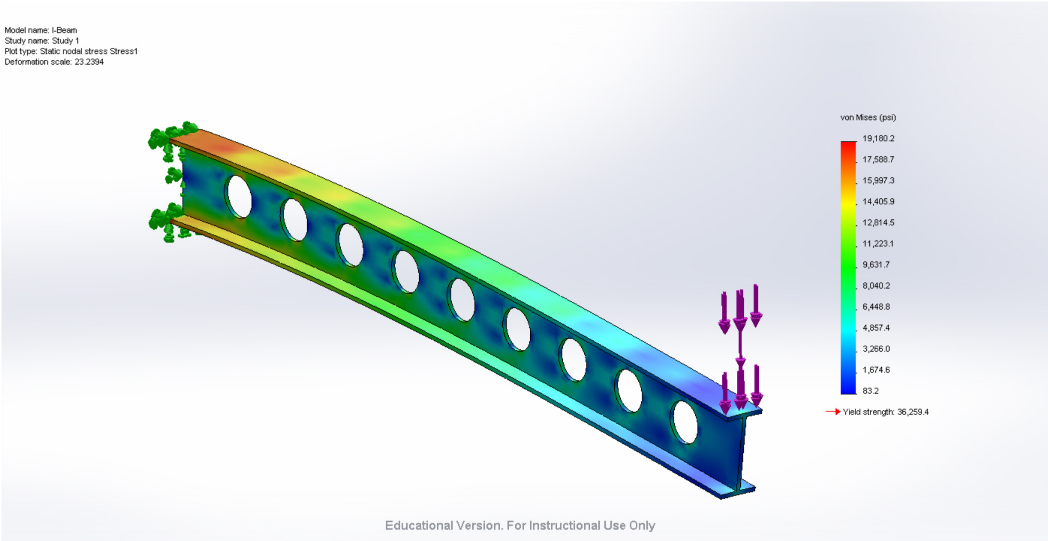


Figure 3. Stress analysis results for the I-beam without holes. The peak stress is just below 20 ksi (red on the scale at right), similar to the value from our simplified analysis.

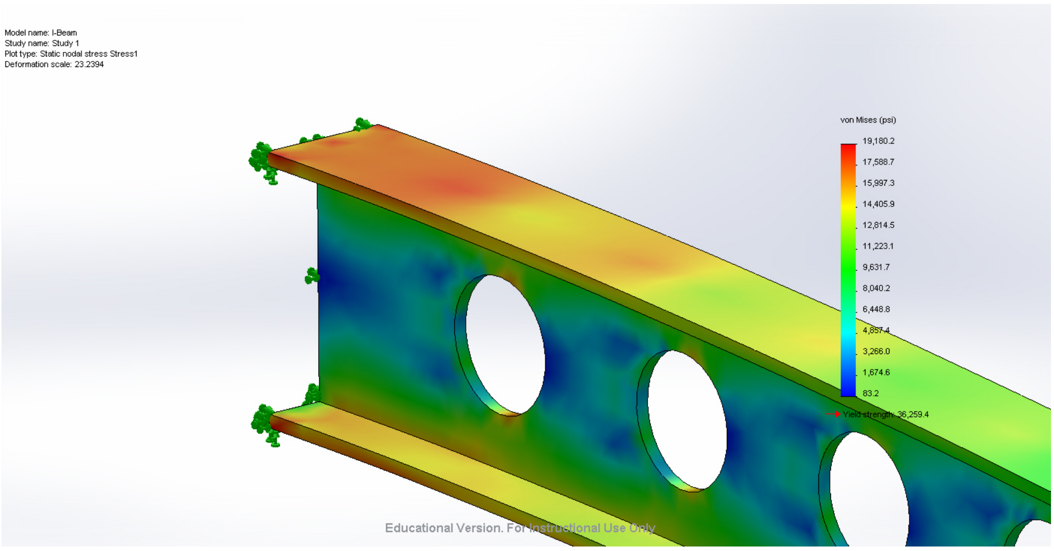


Figure 4. Close-up of the peak stress region for the beam without holes. The peak stress occurs on the top and bottom edges of the flange near the cantilever point, as expected from our simple model.

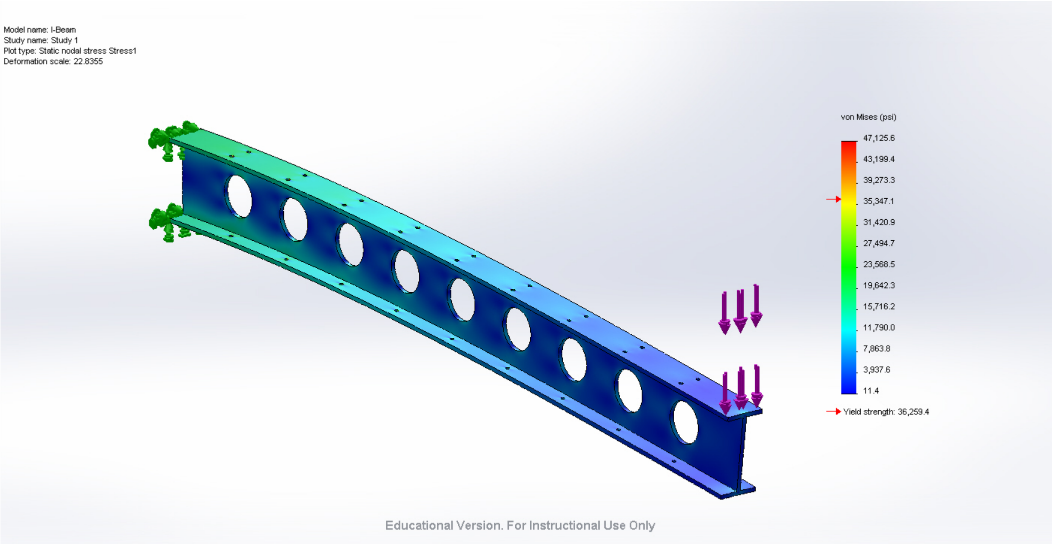


Figure 5. Stress analysis results for the I-beam including holes. The peak stress has increased to above 47 ksi, slightly lower than the 54 ksi predicted by our stress concentration factor analysis. The red arrow on the color legend indicates the expected yield stress for the material we have applied to this part, a near-literal 'red flag' for most design problems.

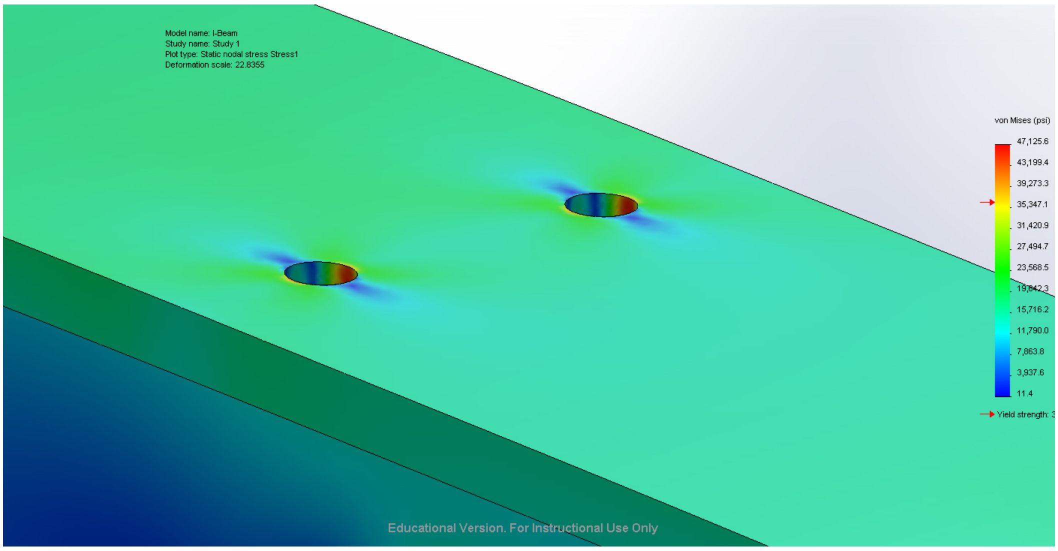


Figure 6. Close-up of the holes on the flange nearest the cantilever. The peak stress in the beam occurs at the edge of these holes, as expected based on our simple analysis of beam bending and stress concentrations.

Reflect and interpret: Note that the results of the detailed stress analysis match nicely with those of our simple analysis, despite the ruthless simplifications we made. But imagine now that we were aiming for a particular factor of safety. We would need to either guess and check iteratively or use a software tool (such as 'design optimization' in SolidWorks) to tune selected parameters in a more structured way. Either of these processes would require significant time for computation and would not provide a deeper understanding of how design parameters interact with outcomes of interest. Could we do better using our analytical models, which seem reasonably accurate at predicting key outcomes?

3.3.3 Design using inverse analysis

Instead of assuming we are given all the parameters of our design, let's consider the case where only some parameters are constrained and we are left to select the best values for remaining parameters according to some desired outcome.

For example, assume the beam must span a given distance (L is fixed), support a given load (F is fixed), and provide mounting holes for preset components (d is fixed and K_t can be approximated). Further, let us for now restrict ourselves to the same material (σ_y is set), and assume a domain with well-understood risks and rewards ($F.O.S.$ is set). Rearranging Equation (3.4) and including the stress concentration factor, we have:

$$b \cdot t \cdot h \approx \frac{F.O.S. \cdot K_t \cdot F \cdot L}{\sigma_y} \tag{3.6}$$

The terms on the left hand side of Equation (3.6) are free design parameters, while those on the right hand side are design requirements. Since we have one equation and three free parameters, this design problem is under-constrained; there are infinite possible solutions that all satisfy our requirements equally well.

What to do? Whenever you have an under-constrained system, it's time to look for more equations or constraints. One possibility is to look to additional well-defined outcomes that interact with our design parameters. A common example would be the total mass of the I-beam. Let's calculate the mass as density times volume, and make simplifying assumptions:

$$m = \rho \cdot L \cdot \left[2bt + t(h - \cancel{2t})\right]$$

$$\text{(again assuming the } t^2 \text{ term will be small)} \tag{3.7}$$

$$m \approx \rho L t(2b + h).$$

Now we can combine Equations 3.6 and 3.7 to reduce the design space. We will have to pick one parameter to substitute, and therefore eliminate, in the new equation. Perhaps in this design, looking at additional outcomes of interest, we might discover that there are good reasons to set b and h to certain values, while setting t *a priori* does not help as much. We might then rearrange as:

$$t \approx \frac{FoS \cdot K_t \cdot F \cdot L}{b \cdot h \cdot \sigma_y} \tag{3.8}$$

We can then combine the equation for an outcome we wish to minimize (mass) with that for an outcome we wish to set (factor of safety) to reduce the design space:

$$m \approx \frac{\rho \cdot FoS \cdot K_t \cdot F \cdot L^2}{\sigma_y} \cdot \frac{2b + h}{bh}, \tag{3.9}$$

The first term on the right hand side of the equation is set by our design constraints, while the second term reveals the relationship between our free design parameters, b and h, and the outcome we wish to minimize, mass. In this case, you can prove to yourself that $(2b+h)/bh$ is minimized as b and h go to infinity. In other words, if we want the beam to be light, we want it to be tall, wide, and (noting that we have implicitly set t by the substituted relationship) thin.

Reflect and interpret: Using inverse analysis, we have quickly generated an understanding, in the form of Equation 3.9, of the relationship between design parameters, design requirements and design objectives to be optimized in this problem. Considering this relationship informs our intuition more quickly and accurately than guessing and checking with a CAD model. The derived relationship also allows for rapid analytical optimization of design parameters, not only valid for one set of parameters but for all variations of this qualitative problem, a feature that is very helpful when iteratively re-designing components, e.g., during design of an assembly. In this way we have taken up the models that we came to think of as static equations for analysis of existing scenarios and repurposed them as powerful tools for developing our intuition and optimizing new designs.

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