Carnegie Mellon
School of Computer Science

Deep Reinforcement Learning and Control

Function Approximation for (on policy) Prediction and Control

Lecture 8, CMU 10-403

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Used Materials

- **Disclaimer**: Much of the material and slides for this lecture were borrowed from Russ Salakhutdinov, Rich Sutton's class and David Silver's class on Reinforcement Learning.

Large-Scale Reinforcement Learning

- Reinforcement learning has been used to solve large problems, e.g.
  - Backgammon: 10^{20} states
  - Computer Go: 10^{170} states
  - Helicopter: continuous state space

- Tabular methods clearly do not work

Value Function Approximation (VFA)

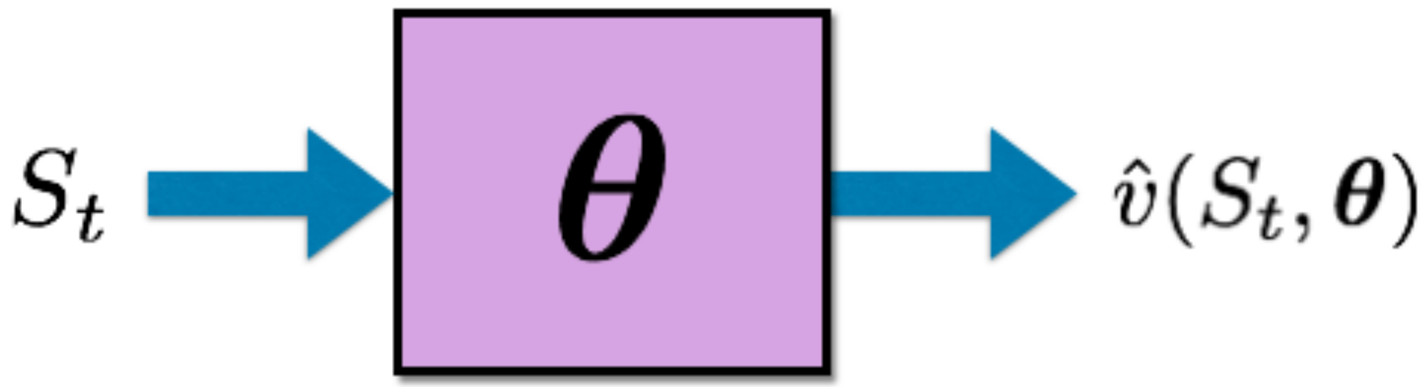
- So far we have represented value function by a lookup table
  - Every state s has an entry V(s), or
  - Every state-action pair (s,a) has an entry Q(s,a)
- Problem with large MDPs:
  - There are too many states and/or actions to store in memory
  - It is too slow to learn the value of each state individually
- Solution for large MDPs:
  - Estimate value function with function approximation

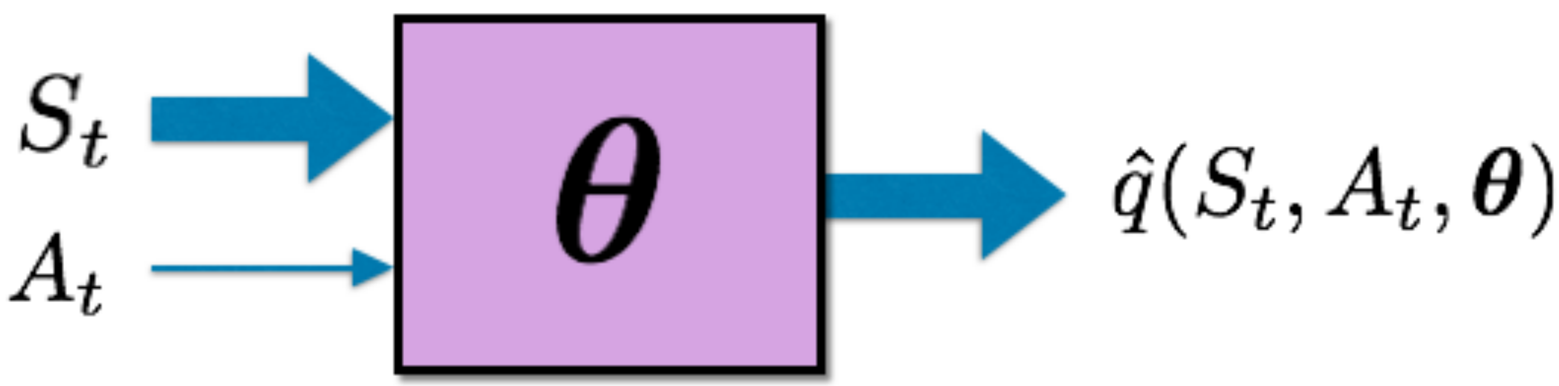
$$\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$$
$$\text{or } \hat{q}(s, a, \mathbf{w}) \approx q_\pi(s, a)$$

  - Generalize from seen states to unseen states

Value Function Approximation (VFA)

- Value function approximation (VFA) replaces the table with a general parameterized form:





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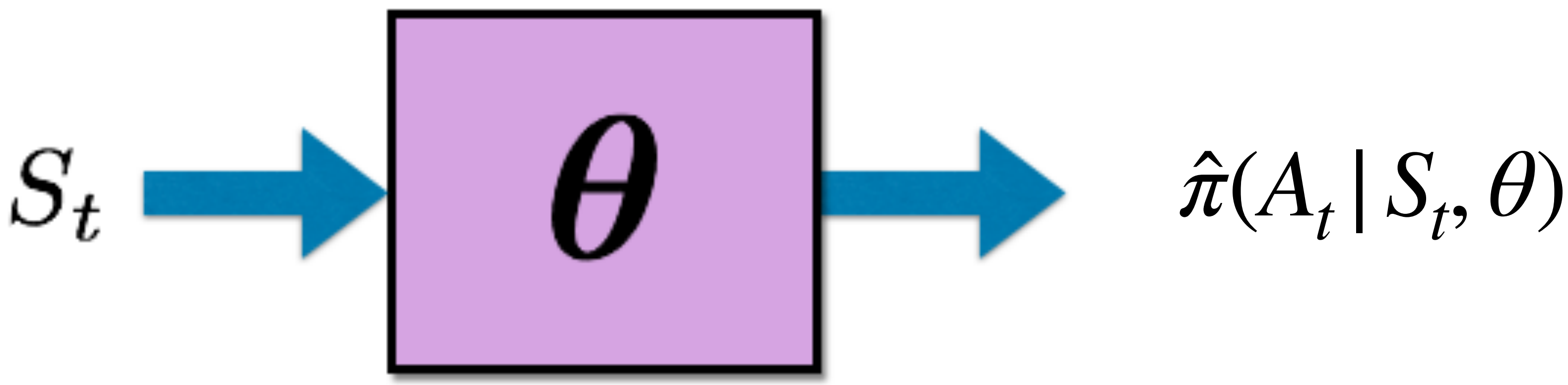

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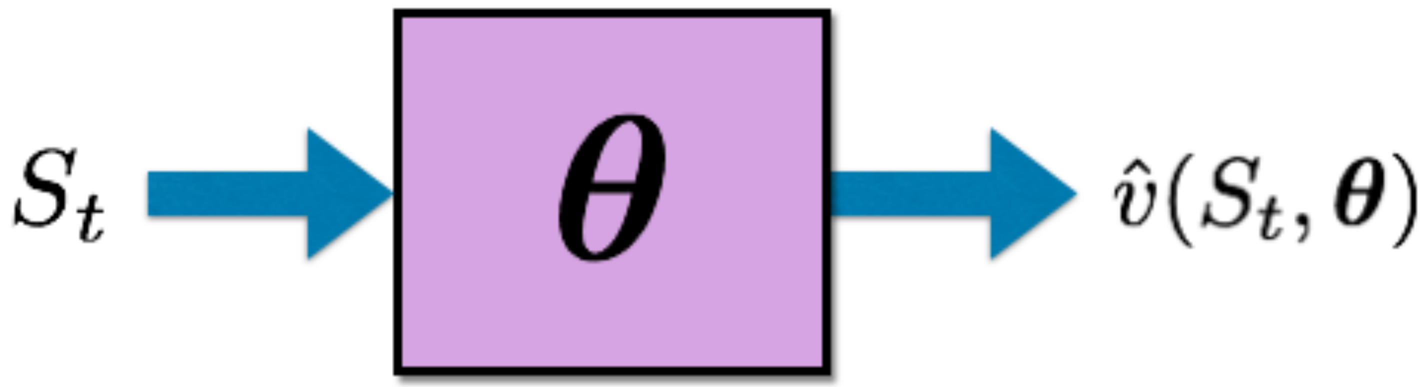
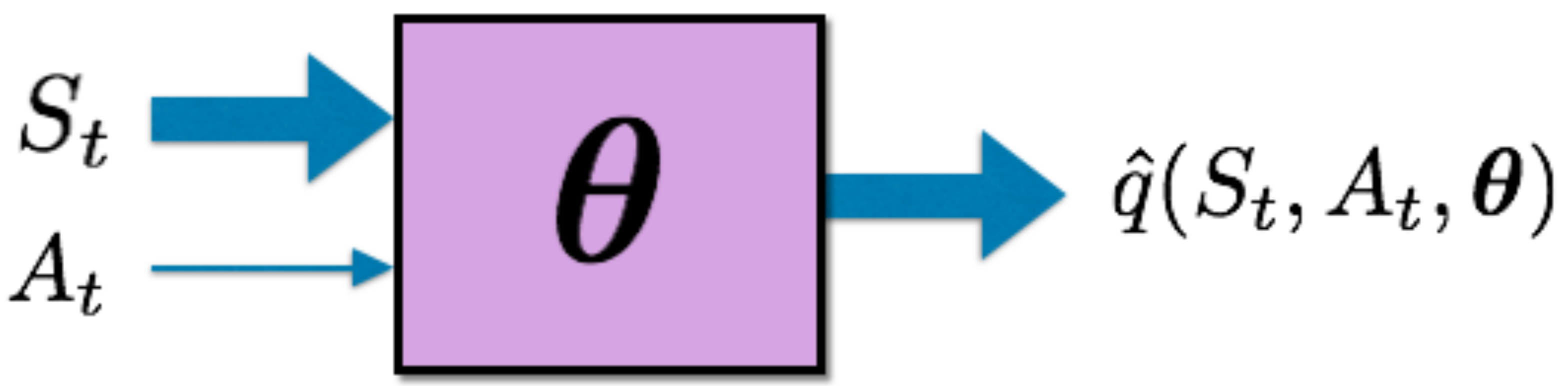


$$|\theta| << |\mathcal{S}|$$



When we update the parameters θ, the values of many states change simultaneously!

Which Function Approximation?

- There are many function approximators, e.g.
  - Linear combinations of features
  - Neural networks
  - Decision tree
  - Nearest neighbour
  - Fourier / wavelet bases
  - …

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- There are many function approximators, e.g.
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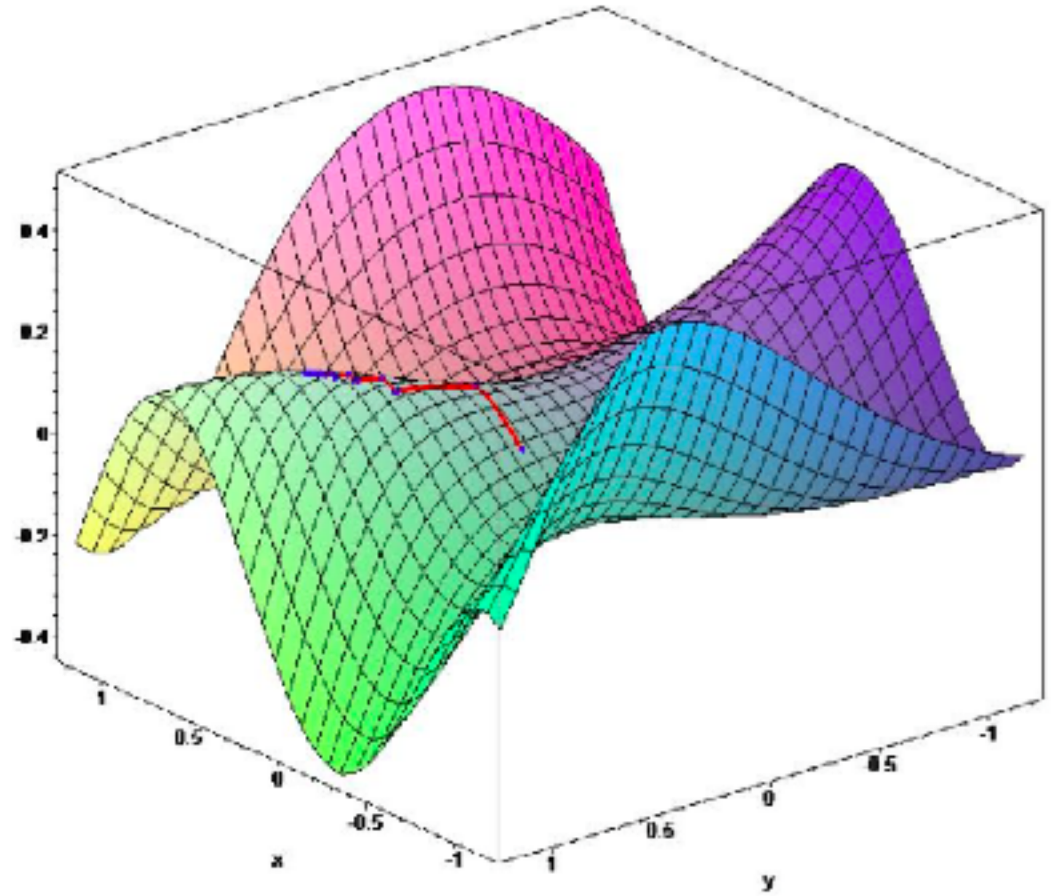
  - differentiable function approximators

Gradient Descent

- Let J(w) be a differentiable function of parameter vector w
- Define the gradient of J(w) to be:

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = \begin{pmatrix} \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_1} \\ \vdots \\ \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_n} \end{pmatrix}$$



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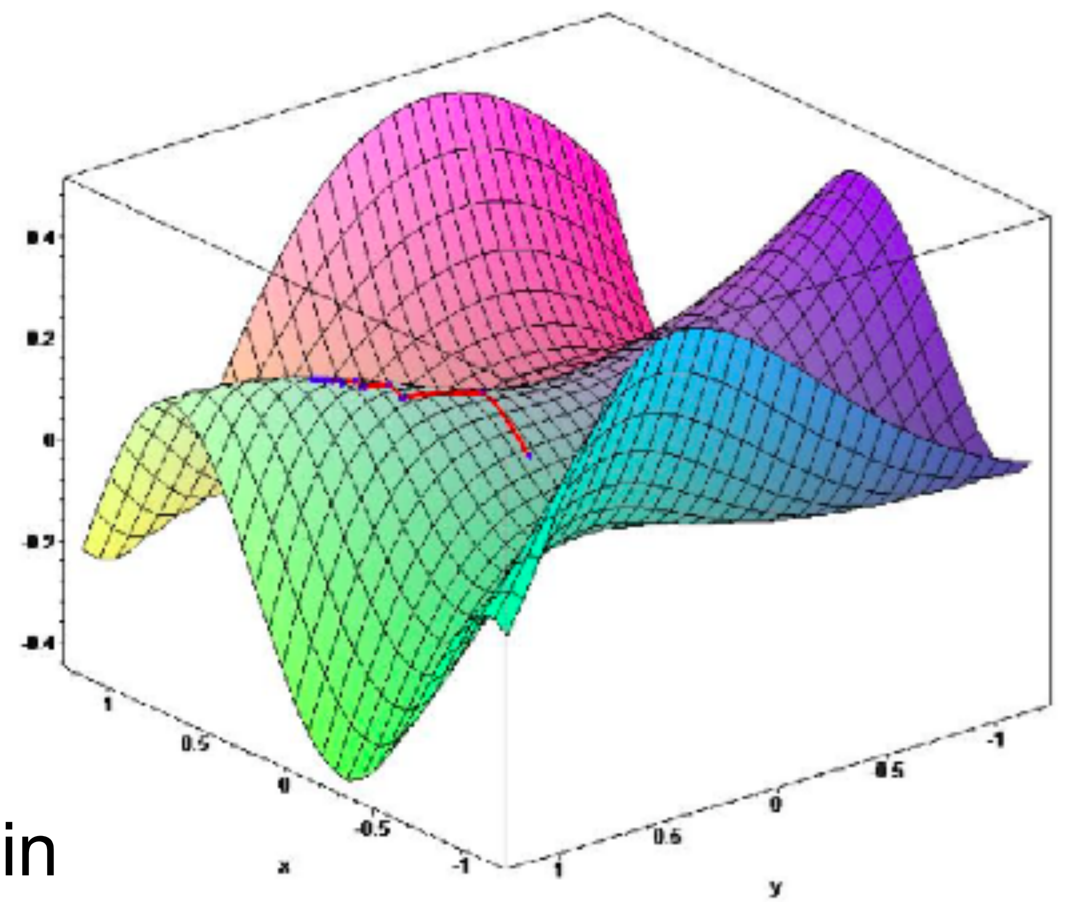

- To find a local minimum of J(w), adjust w in direction of the negative gradient:

$$\Delta \mathbf{w} = -\frac{1}{2}\alpha \nabla_{\mathbf{w}} J(\mathbf{w})$$

Step-size

Gradient Descent

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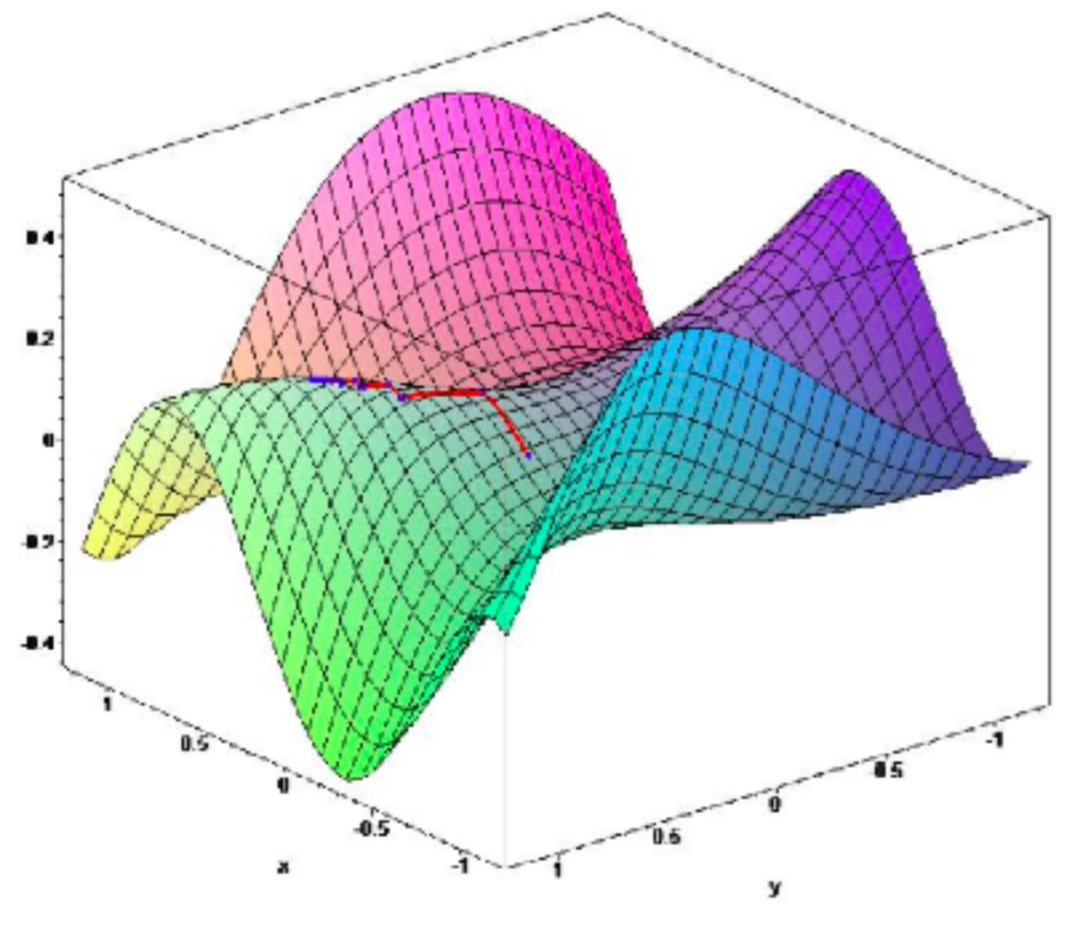

- Starting from a guess $\mathbf{w}_0$
- We consider the sequence $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$ s.t. :

$$\mathbf{w}_{n+1} = \mathbf{w}_n - \frac{1}{2}\alpha \nabla_{\mathbf{w}} J(\mathbf{w}_n)$$

- We then have $J(\mathbf{w}_0) \geq J(\mathbf{w}_1) \geq J(\mathbf{w}_2) \geq \ldots$

Our objective

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$

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$$J(\mathbf{w}) = \mathbb{E}_\pi \left[(v_\pi(S) - \hat{v}(S, \mathbf{w}))^2 \right]$$

Let $\mu(S)$ denote how much time we spend in each state s under policy π, then:

$$J(w) = \sum_{n=1}^{|\mathcal{S}|} \mu(S) \left[v_\pi(S) - \hat{v}(S, \mathbf{w}) \right]^2 \qquad \sum_{s \in \mathcal{S}} \mu(S) = 1$$

Very important choice: it is OK if we cannot learn the value of states we visit very few times, there are too many states, I should focus on the ones that matter: the RL way of approximating the Bellman equations!

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In contrast to:

$$J_2(w) = \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \left[v_\pi(S) - \hat{v}(S, \mathbf{w}) \right]^2$$

On-policy state distribution

Let $h(s)$ be the initial sate distribution, i.e, the probability that an episode starts at state s, then:

$$\eta(s) = h(s) + \sum_{\bar{s}} \eta(\bar{s}) \sum_{a} \pi(a \mid \bar{s}) p(s \mid \bar{s}, a), \; \forall s \in \mathcal{S}$$

$$\mu(s) = \frac{\eta(s)}{\sum_{s'} \eta(s')}, \quad \forall s \in \mathcal{S}$$

Gradient Descent

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$

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$$\begin{aligned} \Delta \mathbf{w} &= -\frac{1}{2} \alpha \nabla_{\mathbf{w}} J(\mathbf{w}) \\ &= \alpha \mathbb{E}_\pi \left[(v_\pi(S) - \hat{v}(S, \mathbf{w})) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) \right] \end{aligned}$$

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- We consider the sequence $w_0, w_1, w_2, \ldots$ s.t. :

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Gradient Descent

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$$J(\mathbf{w}) = \mathbb{E}_\pi \left[(v_\pi(S) - \hat{v}(S, \mathbf{w}))^2 \right]$$

- Gradient descent finds a local minimum:

$$\begin{aligned} \Delta \mathbf{w} &= -\frac{1}{2} \alpha \nabla_\mathbf{w} J(\mathbf{w}) \\ &= \alpha \mathbb{E}_\pi \left[(v_\pi(S) - \hat{v}(S, \mathbf{w})) \nabla_\mathbf{w} \hat{v}(S, \mathbf{w}) \right] \end{aligned}$$

Stochastic Gradient Descent

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$

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- Gradient descent finds a local minimum:

$$\begin{aligned} \Delta \mathbf{w} &= -\frac{1}{2} \alpha \nabla_{\mathbf{w}} J(\mathbf{w}) \\ &= \alpha \mathbb{E}_\pi \left[(v_\pi(S) - \hat{v}(S, \mathbf{w})) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) \right] \end{aligned}$$

- Stochastic gradient descent (SGD) samples the gradient:

$$\Delta \mathbf{w} = \alpha (v_\pi(S) - \hat{v}(S, \mathbf{w})) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w})$$

Least Squares Prediction

- Given value function approximation: $\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$
- And experience D consisting of ⟨state,value⟩ pairs

$$\mathcal{D} = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, ..., \langle s_T, v_T^\pi \rangle\}$$

- Find parameters w that give the best fitting value function v(s,w)?
- Least squares algorithms find parameter vector w minimizing sum-squared error between $v(S_t,w)$ and target values v_t^π:

$$\begin{aligned} LS(\mathbf{w}) &= \sum_{t=1}^{T} (v_t^\pi - \hat{v}(s_t, \mathbf{w}))^2 \\ &= \mathbb{E}_\mathcal{D}\left[(v^\pi - \hat{v}(s, \mathbf{w}))^2\right] \end{aligned}$$

SGD with Experience Replay

- Given experience consisting of ⟨state, value⟩ pairs

$$\mathcal{D} = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, ..., \langle s_T, v_T^\pi \rangle\}$$

- Repeat
  - Sample state, value from experience

$$\langle s, v^\pi \rangle \sim \mathcal{D}$$

  - Apply stochastic gradient descent update

$$\Delta \mathbf{w} = \alpha(v^\pi - \hat{v}(s, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(s, \mathbf{w})$$

- Converges to least squares solution

Feature Vectors

- Represent state by a feature vector

$$\mathbf{x}(S) = \begin{pmatrix} \mathbf{x}_1(S) \\ \vdots \\ \mathbf{x}_n(S) \end{pmatrix}$$

- For example
  - Distance of robot from landmarks
  - Trends in the stock market
  - Piece and pawn configurations in chess

Linear Value Function Approximation (VFA)

- Represent value function by a linear combination of features

$$\hat{v}(S, \mathbf{w}) = \mathbf{x}(S)^\top \mathbf{w} = \sum_{j=1}^{n} \mathbf{x}_j(S)\mathbf{w}_j$$

- Objective function is quadratic in parameters w

$$J(\mathbf{w}) = \mathbb{E}_\pi\left[\left(v_\pi(S) - \mathbf{x}(S)^\top \mathbf{w}\right)^2\right]$$

- Update rule is particularly simple

$$\nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) = \mathbf{x}(S)$$

$$\Delta\mathbf{w} = \alpha(v_\pi(S) - \hat{v}(S, \mathbf{w}))\mathbf{x}(S)$$

- Update = step-size × prediction error × feature value
- Later, we will look at the neural networks as function approximators.

Incremental Prediction Algorithms

- We have assumed the true value function $v_\pi(s)$ is given by a supervisor
- But in RL there is no supervisor, only rewards
- In practice, we substitute a target for $v_\pi(s)$
- For MC, the target is the return G_t

$$\Delta \mathbf{w} = \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}}\hat{v}(S_t, \mathbf{w})$$

- For TD(0), the target is the TD target: $R_{t+1} + \gamma\hat{v}(S_{t+1}, \mathbf{w})$

$$\Delta \mathbf{w} = \alpha(R_{t+1} + \gamma\hat{v}(S_{t+1}, \mathbf{w}) - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}}\hat{v}(S_t, \mathbf{w})$$

Remember $\Delta \mathbf{w} = \alpha(v_\pi(S) - \hat{v}(S, \mathbf{w}))\nabla_{\mathbf{w}}\hat{v}(S, \mathbf{w})$

Monte Carlo with VFA

- Return G_t is an unbiased, noisy sample of true value $v_\pi(S_t)$
- Can therefore apply supervised learning to "training data":

$$\langle S_1, G_1\rangle, \langle S_2, G_2\rangle, \ldots, \langle S_T, G_T\rangle$$

- For example, using linear Monte-Carlo policy evaluation

$$\begin{aligned}\Delta \mathbf{w} &= \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w}) \\ &= \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\mathbf{x}(S_t)\end{aligned}$$

- Monte-Carlo evaluation converges to a local optimum

Monte Carlo with VFA

Gradient Monte Carlo Algorithm for Approximating $\hat{v} \approx v_\pi$

Input: the policy π to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S} \times \mathbb{R}^n \to \mathbb{R}$

Initialize value-function weights $\boldsymbol{\theta}$ as appropriate (e.g., $\boldsymbol{\theta} = \mathbf{0}$)
Repeat forever:
    Generate an episode $S_0, A_0, R_1, S_1, A_1, \ldots, R_T, S_T$ using π
    For $t = 0, 1, \ldots, T-1$:
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \big[G_t - \hat{v}(S_t, \boldsymbol{\theta}) \big] \nabla \hat{v}(S_t, \boldsymbol{\theta})$

TD Learning with VFA

- The TD-target $R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w})$ a biased sample of true value $v_\pi(S_t)$
- Can still apply supervised learning to “training data”:

$$\langle S_1, R_2 + \gamma \hat{v}(S_2, \mathbf{w})\rangle, \langle S_2, R_3 + \gamma \hat{v}(S_3, \mathbf{w})\rangle, ..., \langle S_{T-1}, R_T\rangle$$

- For example, using linear TD(0):

$$\begin{aligned}\Delta \mathbf{w} &= \alpha(R + \gamma \hat{v}(S', \mathbf{w}) - \hat{v}(S, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) \\ &= \alpha \delta \mathbf{x}(S)\end{aligned}$$

We ignore the dependence of the target on w!

We call it semi-gradient methods

TD Learning with VFA

Semi-gradient TD(0) for estimating $\hat{v} \approx v_\pi$

Input: the policy π to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S}^+ \times \mathbb{R}^n \to \mathbb{R}$ such that $\hat{v}(\text{terminal}, \cdot) = 0$

Initialize value-function weights $\boldsymbol{\theta}$ arbitrarily (e.g., $\boldsymbol{\theta} = \mathbf{0}$)
Repeat (for each episode):
    Initialize S
    Repeat (for each step of episode):
        Choose $A \sim \pi(\cdot|S)$
        Take action A, observe R, S'
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\big[R + \gamma\hat{v}(S',\boldsymbol{\theta}) - \hat{v}(S,\boldsymbol{\theta})\big]\nabla\hat{v}(S,\boldsymbol{\theta})$
        $S \leftarrow S'$
    until S' is terminal

Control with VFA

- Policy evaluation Approximate policy evaluation: $\hat{q}(\cdot, \cdot, \mathbf{w}) \approx q_\pi$
- Policy improvement ε-greedy policy improvement

Action-Value Function Approximation

- Approximate the action-value function

$$\hat{q}(S, A, \mathbf{w}) \approx q_\pi(S, A)$$

- Minimize mean-squared error between the true action-value function $q_\pi(S,A)$ and the approximate action-value function:

$$J(\mathbf{w}) = \mathbb{E}_\pi \left[(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))^2 \right]$$

- Use stochastic gradient descent to find a local minimum

$$-\frac{1}{2} \nabla_{\mathbf{w}} J(\mathbf{w}) = (q_\pi(S, A) - \hat{q}(S, A, \mathbf{w})) \nabla_{\mathbf{w}} \hat{q}(S, A, \mathbf{w})$$

$$\Delta \mathbf{w} = \alpha (q_\pi(S, A) - \hat{q}(S, A, \mathbf{w})) \nabla_{\mathbf{w}} \hat{q}(S, A, \mathbf{w})$$

Linear Action-Value Function Approximation

- Represent state and action by a feature vector

$$\mathbf{x}(S,A) = \begin{pmatrix} \mathbf{x}_1(S,A) \\ \vdots \\ \mathbf{x}_n(S,A) \end{pmatrix}$$

- Represent action-value function by linear combination of features

$$\hat{q}(S,A,\mathbf{w}) = \mathbf{x}(S,A)^\top \mathbf{w} = \sum_{j=1}^{n} \mathbf{x}_j(S,A)\mathbf{w}_j$$

- Stochastic gradient descent update

$$\nabla_{\mathbf{w}} \hat{q}(S,A,\mathbf{w}) = \mathbf{x}(S,A)$$

$$\Delta \mathbf{w} = \alpha(q_\pi(S,A) - \hat{q}(S,A,\mathbf{w}))\mathbf{x}(S,A)$$

Incremental Control Algorithms

- Like prediction, we must substitute a target for $q_\pi(S,A)$
- For MC, the target is the return G_t

$$\Delta \mathbf{w} = \alpha(G_t - \hat{q}(S_t, A_t, \mathbf{w}))\nabla_{\mathbf{w}}\hat{q}(S_t, A_t, \mathbf{w})$$

- For TD(0), the target is the TD target: $R_{t+1} + \gamma Q(S_{t+1}, A_{t+1})$

$$\Delta \mathbf{w} = \alpha(R_{t+1} + \gamma \hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}))\nabla_{\mathbf{w}}\hat{q}(S_t, A_t, \mathbf{w})$$

Can we guess the deep Q learning update rule?

$$\Delta \mathbf{w} = \alpha(R_{t+1} + \gamma \max_{A_{t+1}} \hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}))\nabla_{\mathbf{w}}\hat{q}(S_t, A_t, \mathbf{w})$$

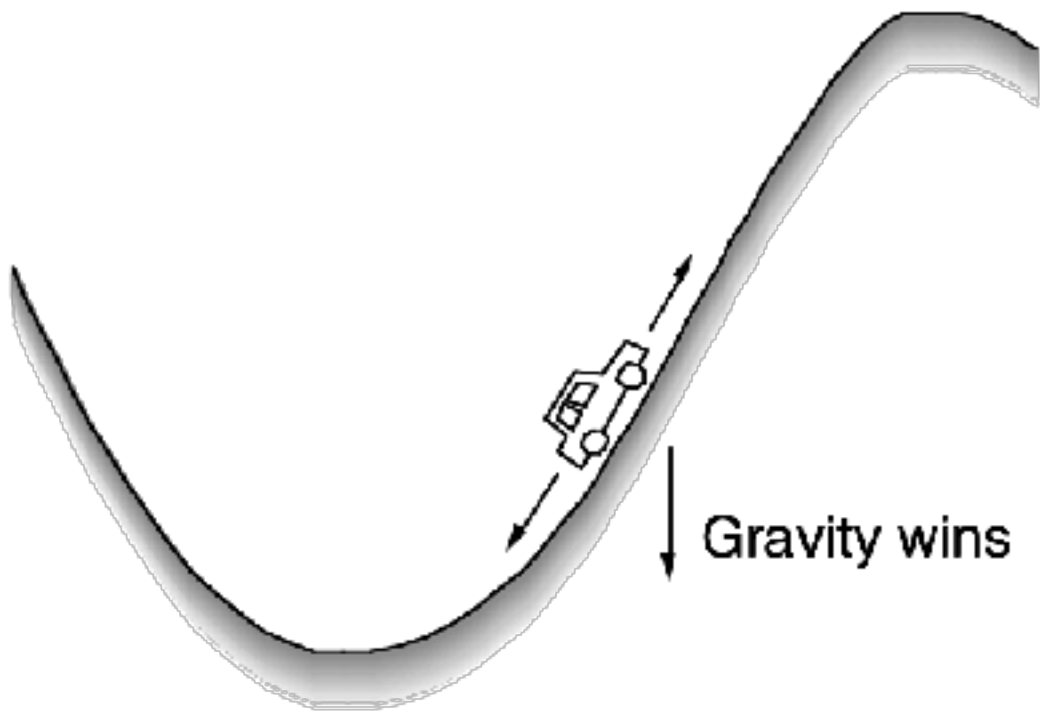
Incremental Control Algorithms

Episodic Semi-gradient Sarsa for Estimating $\hat{q} \approx q_*$

Input: a differentiable function $\hat{q} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^n \to \mathbb{R}$

Initialize value-function weights $\boldsymbol{\theta} \in \mathbb{R}^n$ arbitrarily (e.g., $\boldsymbol{\theta} = \mathbf{0}$)
Repeat (for each episode):
  $S, A \leftarrow$ initial state and action of episode (e.g., ε-greedy)
  Repeat (for each step of episode):
    Take action A, observe R, S'
    If S' is terminal:
      $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\big[R - \hat{q}(S, A, \boldsymbol{\theta})\big]\nabla\hat{q}(S, A, \boldsymbol{\theta})$
      Go to next episode
    Choose A' as a function of $\hat{q}(S', \cdot, \boldsymbol{\theta})$ (e.g., ε-greedy)
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\big[R + \gamma\hat{q}(S', A', \boldsymbol{\theta}) - \hat{q}(S, A, \boldsymbol{\theta})\big]\nabla\hat{q}(S, A, \boldsymbol{\theta})$
    $S \leftarrow S'$
    $A \leftarrow A'$

Example: The Mountain-Car problem



Minimum-Time-to-Goal Problem

<u>SITUATIONS</u>:
car's position and velocity

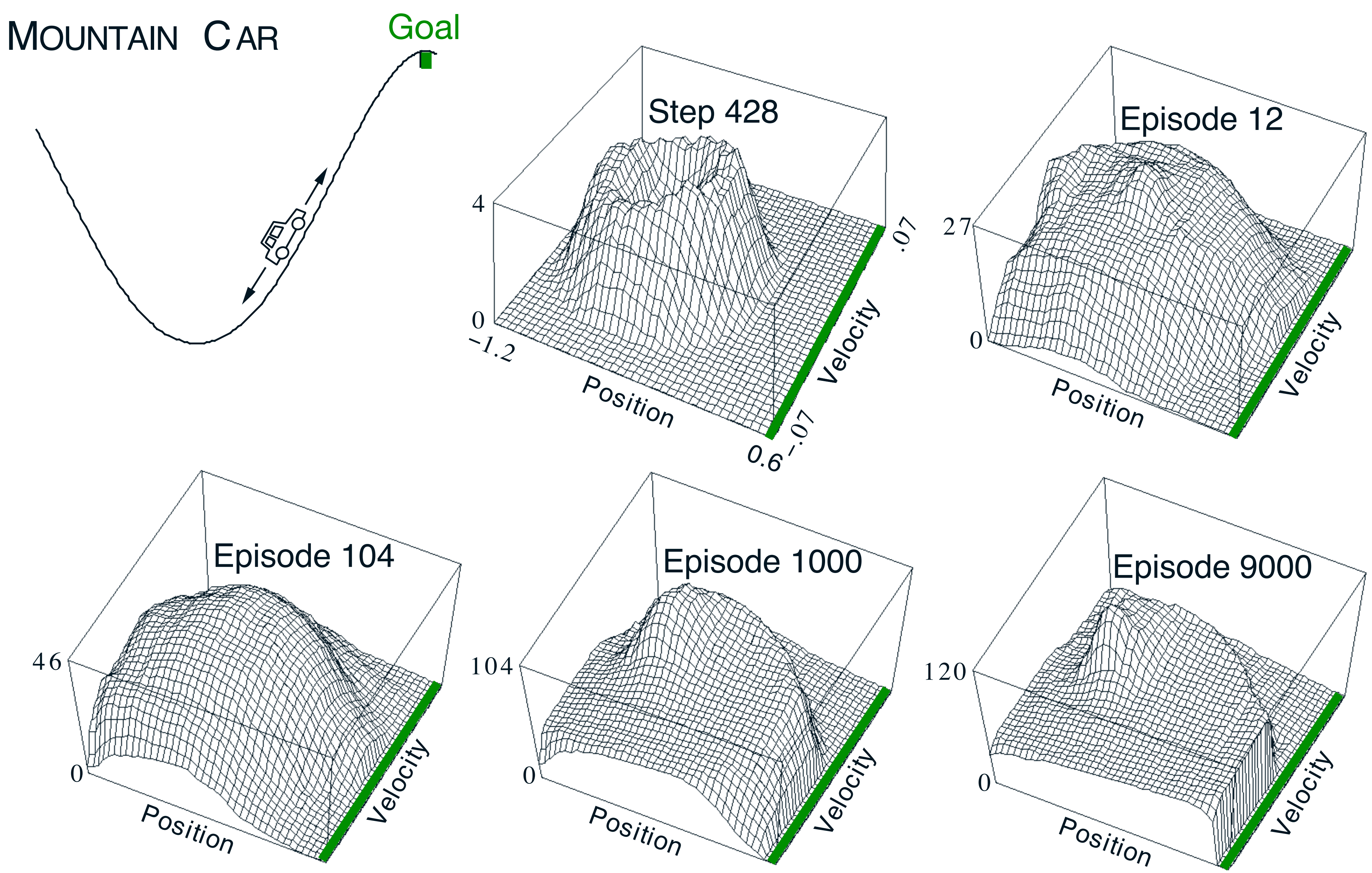
<u>ACTIONS</u>:
three thrusts: forward, reverse, none

<u>REWARDS</u>:
always –1 until car reaches the goal

Episodic, No Discounting, γ=1

Example: The Mountain-Car problem

$$-\max_a \hat{q}(s, a, \boldsymbol{\theta})$$



Batch Reinforcement Learning

- Gradient descent is simple and appealing
- But it is not sample efficient
- Batch methods seek to find the best fitting value function
- Given the agent's experience ("training data")

Which Function Approximation?

- There are many function approximators, e.g.
  - Linear combinations of features
  - Neural networks
  - Decision tree
  - Nearest neighbour
  - Fourier / wavelet bases
  - …

Nearest neighbors

- Save training examples in memory as they arrive (s,v(s)). (state, value)
- Then, given a new state s’, retrieve closest state examples from the memory and average their values based on similarity:

$$v(s') = \sum_{i=1}^{K} k(h_{s'}, h_{s_i}) v(s_i)$$

- Accuracy improves as more data accumulates.
- Agent’s experience has an **immediate affect** on value estimates in the neighborhood of its environment’s current state.
- Parametric methods need to incrementally adjust parameters of a global approximation.