Selling with network flows: a new approach to optimal multi-dimensional mechanisms*

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March 2024

Abstract

We study the classical problem of designing optimal menus when a monopolist sells multiple objects and faces multi-dimensional private information. Using a duality result, we show that solving the relaxed problem of imposing local incentive compatibility constraints is equivalent to finding a set of Lagrange multipliers that yield appropriate signs of virtual values in certain regions of the type space. We then use a network flow approach a la Gale (1957) to show that the existence of such Lagrange multipliers is reduced to the verification of a class of inequalities. We apply this method to a range of problems and examples including optimal selling mechanisms with correlated values.

Extended Abstract

The optimal screening problems have been an important subject of economics research for multiple decades. Its analysis has been successfully applied to many important topics of nonlinear pricing (Wilson, 1993), public good provision (Green and Laffont, 1977), regulation (Baron and Myerson, 1982), taxation (Mirrlees, 1971), auctions (Myerson, 1981), etc. Many of these applications assume one-dimensional parameterization of agent preferences. Even when multi-dimensional types are critical to analyze some economical trade-offs, this assumption is done out of necessity as multi-dimensional incentive compatibility constraints are much harder to analyze. The issue is that the second-order conditions are

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typically binding in multi-dimensional environments, which requires to perform complicated "ironing" or "sweeping" procedures to characterize optimal outcomes Rochet and Stole (2003). In this paper, we identify a range of optimal multi-dimensional screening problems that can be handled with the appropriate first-order approach without necessity to check the second-order conditions.

We use a novel way to relax multi-dimensional incentive compatibility constraints by considering only two paths of integration, which is arguably a minimal departure from the classical Myerson's approach that work so well in single-dimensional environments.[1] This is in contrast to the approaches of some recent papers (e.g. Haghpanah and Hartline, 2020) that aim to find one path of integration where incentive-compatibility constraints are binding. Analyzing two paths of integration reduces the need to search for the right path of integration where incentive constraints are binding.

Using the novel relaxed problem, we formulate its dual in the space of integrable allocation functions. We then exploit the standard techniques in infinite-dimensional spaces Anderson and Nash (1987) to establish no duality gap result. This is in contrast to recent literature that works in the space of continuous agent utility functions that requires to develop smart and often complicated techniques to establish the no duality gap result (Rochet and Choné, 1998; Daskalakis, Deckelbaum, and Tzamos, 2017; Manelli and Vincent, 2006; Cai, Devanur, and Weinberg, 2016; Carroll, 2017; Kolesnikov, Sandomirskiy, Tsyvinski, and Zimin, 2023).

The dual problem reduces the search for a solution of the relaxed problem to identifying Lagrange multipliers that yield appropriate signs of "multi-dimensional" virtual values for goods in certain regions of the type space. Using the expression of virtual values we derive the necessary conditions for the optimal allocation regions that were originally obtained by Manelli and Vincent (2006). We supplement these conditions with the new ones to obtain the necessary and sufficient conditions for allocation regions to satisfy in order to be a part of a revenue-maximizing mechanism. To accomplish this, we discretize the type space and interpret the virtual values as a flow in a certain network. Using the results from the seminal network-flow literature Gale (1957) on the existence of a flow in a network that satisfies the boundary constraints, we show that the existence of optimal Lagrange multipliers that deliver appropriate signs to virtual values is reduced to the verification of a class of inequalities.

Using the novel necessary and sufficient conditions, we were able to provide a sim-

[1] We require that the integral of an allocation rule over two paths connecting any given type with the lowest type in the type space are the same instead of requiring that the integrals over all piece-wise smooth paths result in the same value.

ple criterion that determines whether an optimal mechanism involves randomization or not. This criterion is expressed in terms of the monotonicity of some function that can be derived from the model primitives. If this function is *weakly increasing* the optimal mechanism is *deterministic*, and if it is *decreasing* the optimal mechanism involves *randomization or grand-bundling*. We use this criterion to provide a guide for researchers on how to determine optimal mechanism for a wide range of distributions. We illustrate the guide using numerous examples.

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