Change of Coordinates

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It is often useful to change from one coordinate system to another. For example, between polar and cartesian coordinates for the location of a point in a plane. In robotics, we may want to change between controlling the individual joints of a robot or its end effector location. When and how can we do this?

1 General Case

Consider a system with state $q \in \mathcal{Q}$ that we would instead like to represent with alternate coordinates $y \in \mathcal{Y}$. If we know how to convert our q coordinates into y coordinates through a function $\psi : q \mapsto y$, how do we map velocities, accelerations, and forces? What properties of ψ do we need, and when will ψ^{-1} exist?

1.1 Conversions

First, for any map between two manifolds we know that the tangent map (a.k.a. the differential, the Jacobian, the matrix of partial derivatives) will map velocities,

$$\boxed{y = \psi(q) \Rightarrow \dot{y} = Y\dot{q} \quad \text{where} \quad Y(q) = D\psi(q)} \tag{1}$$

The matrix $Y : T\mathcal{Q} \to T\mathcal{Y}$ is in general a function of state q. From this, we can also consider how accelerations map,

$$\boxed{\ddot{y} = Y\ddot{q} + \dot{Y}\dot{q}} \tag{2}$$

Interestingly, the same matrix Y transposed will map forces on our new coordinate system τ_y onto forces in our original coordinate system τ_q,

$$\boxed{\dot{y} = Y\dot{q} \Leftrightarrow \tau_q = Y^T\tau_y} \tag{3}$$

To see why this must be true, consider the power as calculated in each coordinate system,

$$\tau_q^T\dot{q} = (Y^T\tau_y)^T\dot{q} = \tau_y^T Y\dot{q} = \tau_y^T\dot{y} \tag{4}$$

As we can see, the power is the same whether we calculate it in the q system or the y.

If there is a velocity constraint on our alternate system, $\tilde{A}\dot{y} = 0$, we can find the equivalent constraint A on the original coordinate system as,

$$0 = \tilde{A}(y)\dot{y} \tag{5}$$

$$0 = \tilde{A}(\psi(q))Y(q)\dot{q} = A(q)\dot{q} \tag{6}$$

$$A(q) := \tilde{A}(\psi(q))Y(q) \tag{7}$$

To convert back from y to q, we must assume that the inverse function $\psi^{-1}(y) = q$ is well defined and differentiable. In that case, we have,

$$\boxed{q = \psi^{-1}(y) \Rightarrow \dot{q} = H\dot{y} \quad \text{where} \quad H(y) = D\psi^{-1}(y) = Y^{-1}(\psi^{-1}(y))} \tag{8}$$

and similarly,

$$\boxed{\ddot{q} = H\ddot{y} + \dot{H}\dot{y}} \tag{9}$$

$$\boxed{\tau_y = H^T \tau_q} \tag{10}$$

If we have a constraint $A\dot{q} = 0$ on the original system, then we can find the constraint $\tilde{A}$ on the alternate system,

$$0 = A(q)\dot{q} \tag{11}$$

$$0 = A(\psi^{-1}(y))H(y)\dot{y} = \tilde{A}(y)\dot{y} \tag{12}$$

$$\tilde{A}(y) := A(\psi^{-1}(y))H(y) \tag{13}$$

1.2 Existence

When is the mapping from coordinates q to y allowable? We can always apply any arbitrary function ψ to q but that does not mean that we will get something useful. Here, we will say that a change of coordinate is well defined if the new coordinate system still represents the system without any errors or losses. Thus, we should be able to recreate our original state, i.e. we need the inverse ψ^{-1} to map us back to the same q that we started with (and similarly for velocity, etc). For this, we use the inverse function theorem,

Theorem 1 (Inverse Function Theorem)**.** *If a function ψ is continuously differentiable in a neighborhood of (open set around) a point q, i.e. $Y = D\psi$ exists everywhere in that open set and it is a continuous function of q, and if that differential Y is invertible over an open subset of that neighborhood, then the inverse map ψ^{-1} exists locally and its derivative is the matrix inverse Y^{-1}.*

So there are several conditions we need to meet here. First, the function ψ must be continuously differentiable, so it cannot have any discontinuities or points where the derivative is not well defined. Thus coordinate maps like $y = \tan(q)$ or $y = 1/q^2$ will cause problems at certain points.

Second, the matrix Y must be invertible. That means it must be full rank. Thus we cannot have coordinate transforms where directions of Y become singular. A transform of $y = \cos(q)$ will have a differential $Y = [-\sin(q)]$, which is singular at $q = 0$ (as we cannot take $Y^{-1} = [0]^{-1}$). Physically, the problem here is that if we have a velocity from $q = 0$ in either the positive or negative direction we get the same change in y.

The inverse function theorem tells us that these two conditions are sufficient locally, but that the same change of coordinate function ψ may not be well defined everywhere. In that case, we can use a collection of different coordinate transforms to cover the space. However, if we want a globally well defined coordinate change, we need the ψ function to map to every point $y \in \mathcal{Y}$ (i.e. it is surjective or onto) and that each point y is the image of only a single point q (i.e. it is injective or one-to-one). Together, these conditions mean the function is bijective and therefore invertible, so we have a well defined and unique ψ^{-1} everywhere and that the inverse mapping of velocities will be $\dot{q} = Y^{-1}\dot{y}$, where the matrix inverse of Y is also the differential $D\psi^{-1}$. For our dynamics problems, we will also want to reason about velocity and acceleration, so we will further require ψ to be twice differentiable (i.e. ψ is in C^2).

$$\boxed{\psi : q \mapsto y \text{ is a well defined change of coordinates globally if it a bijection and twice differentiable.}} \tag{14}$$

1.3 Example

The first change of coordinates that many of us learn in school is Cartesian to polar coordinates. Let $q = \begin{bmatrix} q_x & q_y \end{bmatrix}^T$ be our Cartesian coordinates and $y = \begin{bmatrix} y_r & y_\theta \end{bmatrix}^T$ be our polar coordinates. Then,

$$\psi(q) = \begin{bmatrix} \sqrt{q_x^2 + q_y^2} \\ \tan^{-1} \frac{q_y}{q_x} \end{bmatrix} = \begin{bmatrix} y_r \\ y_\theta \end{bmatrix} = y \tag{15}$$

$$Y = D\psi(q) = \begin{bmatrix} \frac{q_x}{\sqrt{q_x^2+q_y^2}} & \frac{q_y}{\sqrt{q_x^2+q_y^2}} \\ \frac{-q_y}{q_x^2+q_y^2} & \frac{q_x}{q_x^2+q_y^2} \end{bmatrix} \tag{16}$$

This change of coordinates meets our existence requirements almost everywhere – it is singular at $q = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. But other than the origin, we can use either coordinate system to describe our system.

We can use this change of coordiantes to calculate $\dot{y}_r$ and $\dot{y}_\theta$. For example, if we are at the point $q = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, a vertical velocity of $\dot{q} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ results in a velocity of $\dot{y}_\theta = 1$ as,

$$\dot{y} = \begin{bmatrix} \frac{q_x}{\sqrt{q_x^2+q_y^2}} & \frac{q_y}{\sqrt{q_x^2+q_y^2}} \\ \frac{-q_y}{q_x^2+q_y^2} & \frac{q_x}{q_x^2+q_y^2} \end{bmatrix} \dot{q} = \begin{bmatrix} \frac{1}{\sqrt{1^2+0^2}} & \frac{0}{\sqrt{1^2+0^2}} \\ \frac{-0}{1^2+0^2} & \frac{1}{1^2+0^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{17}$$

If instead we were at the point $q = \begin{bmatrix} 2 & 0 \end{bmatrix}^T$, the velocity becomes $\dot{y}_\theta = 1/2$,

$$\dot{y} = \begin{bmatrix} \frac{q_x}{\sqrt{q_x^2+q_y^2}} & \frac{q_y}{\sqrt{q_x^2+q_y^2}} \\ \frac{-q_y}{q_x^2+q_y^2} & \frac{q_x}{q_x^2+q_y^2} \end{bmatrix} \dot{q} = \begin{bmatrix} \frac{2}{\sqrt{2^2+0^2}} & \frac{0}{\sqrt{2^2+0^2}} \\ \frac{-0}{2^2+0^2} & \frac{2}{2^2+0^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \tag{18}$$

as our change in angle y_θ is smaller the further out we go.

The inverse mapping is then,

$$\psi^{-1}(y) = \begin{bmatrix} y_r \cos(y_\theta) \\ y_r \sin(y_\theta) \end{bmatrix} = \begin{bmatrix} q_x \\ q_y \end{bmatrix} = q \tag{19}$$

$$D\psi^{-1}(y) = \begin{bmatrix} \cos(y_\theta) & -y_r \sin(y_\theta) \\ \sin(y_\theta) & y_r \cos(y_\theta) \end{bmatrix} \tag{20}$$

It does not at first glance look like $D\psi^{-1}$ is the matrix inverse of Y in (16), as stated in the inverse function theorem. However, note that we have written these two matrices in terms of our different coordinates, and we must substitute, using ψ or ψ^{-1}, into the same coordinates in order to compare,

$$Y(\psi^{-1}(y)) = \begin{bmatrix} \frac{y_r \cos(y_\theta)}{\sqrt{(y_r \cos(y_\theta))^2 + (y_r \sin(y_\theta))^2}} & \frac{y_r \sin(y_\theta)}{\sqrt{(y_r \cos(y_\theta))^2 + (y_r \sin(y_\theta))^2}} \\ \frac{-y_r \sin(y_\theta)}{(y_r \cos(y_\theta))^2 + (y_r \sin(y_\theta))^2} & \frac{y_r \cos(y_\theta)}{(y_r \cos(y_\theta))^2 + (y_r \sin(y_\theta))^2} \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} \frac{y_r \cos(y_\theta)}{y_r} & \frac{y_r \sin(y_\theta)}{y_r} \\ \frac{-y_r \sin(y_\theta)}{y_r^2} & \frac{y_r \cos(y_\theta)}{y_r^2} \end{bmatrix} = \begin{bmatrix} \cos(y_\theta) & \sin(y_\theta) \\ \frac{-\sin(y_\theta)}{y_r} & \frac{\cos(y_\theta)}{y_r} \end{bmatrix} \tag{22}$$

Then we can easily check that this expression of Y is indeed the matrix inverse of $D\psi^{-1}$.

2 Eliminating Constraints

For constrained systems, we can also perform a change of coordinates to a reduced coordinate system that eliminates the constraints from the system [1, Sec. II-G]. For example, a simple rigid pendulum could be represented with a state vector $q = \begin{bmatrix} q_x & q_y \end{bmatrix}^T$, storing its x and y location, plus an additional constraint that the length of the pendulum is constant. But we can also represent the system by a single variable θ that does not have any constraints.

2.1 Derivation

Consider a system with n variables and m constraints of the form $A\dot{q} = 0$, where A is an $m \times n$ matrix. We can choose a set of $e = n - m$ reduced coordinates y that are unconstrained through the change of coordinates $y = \psi(q)$. This could be a subset of the coordinates in q or it could be a different set of coordinates. As before, let $Y = D\psi(q)$ be an $e \times n$ matrix such that $\dot{y} = Y\dot{q}$.

Note that ψ, and therefore Y, are a many-to-one mapping (not square) and therefore cannot satisfy the general change of coordinate conditions (neither ψ nor Y are invertible). However, we will still be able to use these reduced coordinates to represent our system and reconstruct our state by considering the constraints.

We have two different functions of $\dot{q}$,

$$A\dot{q} = 0 \tag{23}$$

$$Y\dot{q} = \dot{y} \tag{24}$$

Factoring out the leading matrices and solving,

$$\begin{bmatrix} A \\ Y \end{bmatrix} \dot{q} = \begin{bmatrix} 0_{m\times e} \\ Id_{e\times e} \end{bmatrix} \dot{y} \tag{25}$$

If we have chosen a good ψ such that the leading matrix is invertible, then we see that,

$$\dot{q} = \left(\begin{bmatrix} A \\ Y \end{bmatrix}^{-1} \begin{bmatrix} 0_{m\times e} \\ Id_{e\times e} \end{bmatrix} \right) \dot{y} \tag{26}$$

where this combined term is an $n \times e$ matrix that we will call H such that[1],

$$\boxed{\dot{q} = H\dot{y} \quad \text{where} \quad H = \begin{bmatrix} A \\ Y \end{bmatrix}^{-1} \begin{bmatrix} 0_{m\times e} \\ Id_{e\times e} \end{bmatrix}} \tag{27}$$

This H will act just like the H in (8), but with this alternate definition in this reduced coordinate case.

As a common example, if the coordinate change ψ is a subset of the identity map, such that $y = \psi(q)$ returns the last e elements of q, we get,

$$A\dot{q} = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = 0 \tag{28}$$

$$Y\dot{q} = \begin{bmatrix} 0 & Id_{e\times e} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \dot{y} \tag{29}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & Id_{e\times e} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ Id_{e\times e} \end{bmatrix} \dot{y} = \begin{bmatrix} -A_1^{-1}A_2 \\ Id_{e\times e} \end{bmatrix} \dot{y} \tag{30}$$

$$H = \begin{bmatrix} -A_1^{-1}A_2 \\ Id_{e\times e} \end{bmatrix} \tag{31}$$

For constrained systems, we will say that the change of coordinates is well defined if the matrix inverse in (27) is well defined, that is the rows of A and Y are independent. Thus, we cannot define a constrained direction as our

[1] Note that the order of the equations in (25) used to construct H does not matter, as,

$$H = \begin{bmatrix} Y \\ A \end{bmatrix}^{-1} \begin{bmatrix} I_{e\times e} \\ 0_{m\times e} \end{bmatrix} = \left(\begin{bmatrix} 0_{e\times m} & I_{e\times e} \\ I_{m\times m} & 0_{m\times e} \end{bmatrix} \begin{bmatrix} A \\ Y \end{bmatrix} \right)^{-1} \begin{bmatrix} I_{e\times e} \\ 0_{m\times e} \end{bmatrix} = \begin{bmatrix} A \\ Y \end{bmatrix}^{-1} \begin{bmatrix} 0_{m\times e} & I_{m\times m} \\ I_{e\times e} & 0_{e\times m} \end{bmatrix} \begin{bmatrix} I_{e\times e} \\ 0_{m\times e} \end{bmatrix} = \begin{bmatrix} A \\ Y \end{bmatrix}^{-1} \begin{bmatrix} 0_{m\times e} \\ I_{e\times e} \end{bmatrix}$$

coordinates in the reduced space. If we also want to consider dynamics, we will further require the derivative of H, and therefore the derivative of A and Y, to also be well defined so that we can map accelerations as well,

$$\ddot{y} = Y\ddot{q} + \dot{Y}\dot{q} \tag{32}$$
$$\ddot{q} = H\ddot{y} + \dot{H}\dot{y} \tag{33}$$

What is $\dot{H}$ in terms of A and Y? We can take the derivative directly or use the identity

$$\dot{H} = \begin{bmatrix} A \\ Y \end{bmatrix}^{-1} \begin{bmatrix} -\dot{A} \\ -\dot{Y} \end{bmatrix} H \tag{34}$$

2.2 Properties

With H, we can take any value of $\dot{y}$ and reconstruct our velocity in our original coordinates $\dot{q} = H\dot{y}$. Recall that Y was a many-to-one mapping, so which value of $\dot{q}$ do we get from H? We will get the velocity that agrees with the constraints, $A\dot{q} = 0$, as that equation was used to define H. Thus,

$$AH\dot{y} = 0 \;\forall\, \dot{y} \qquad \Rightarrow \qquad \boxed{AH = 0_{m\times e}} \tag{35}$$

Compare this result to that of (13) – here, there are no constraints on $\dot{y}$, so $\tilde{A} = 0$.

H is no longer a simple inverse of Y, but taking the output of $H\dot{y}$ and mapping it back through Y will get the original value of $\dot{y}$. However, the reverse is not true for $\dot{q}$ that do not satisfy the constraints,

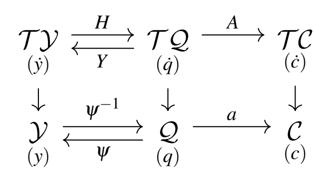


Figure 1: Functional mapping for change of coordinates. Note that this diagram commutes only on the constrained subspace.

$$\boxed{YH = Id_{e\times e}}, \qquad HY \neq Id_{n\times n} \text{ however } \boxed{\forall \dot{q} : A\dot{q} = 0, HY\dot{q} = \dot{q}} \tag{36}$$

In particular, the product HY will have rank $e < n$, as it will zero out each of the m constrained directions.

2.3 Example

Consider a simple pendulum of length l pivoting about the origin. We can store the state of the system as $q = \begin{bmatrix} q_x & q_y \end{bmatrix}^T$, representing the x and y position of the mass. But then we need a constraint,

$$a(q_x, q_y) = q_x^2 + q_y^2 - l^2 = 0 \tag{37}$$
$$A = Da = \begin{bmatrix} 2q_x & 2q_y \end{bmatrix} \tag{38}$$

If we want to eliminate this constraint, we can use a change of coordinates. If we choose the height of the pendulum, $y = \psi(q) = \begin{bmatrix} q_y \end{bmatrix}$, such that $n = 2$, $m = 1$, and $e = 1$, we get the following change of coordinates,

$$Y = D\psi = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{39}$$
$$H = \begin{bmatrix} A \\ Y \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2q_x & 2q_y \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{40}$$
$$= \frac{1}{2q_x} \begin{bmatrix} 1 & -2q_y \\ 0 & 2q_x \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-q_y}{q_x} \\ 1 \end{bmatrix} \tag{41}$$

To check our intuition, we see that the second term of H is 1, as q_y is the same in both q and y. We can also easily verify the properties that $AH = 0$ and $YH = 1$.

When is this a good change of coordinates? We can see that there will be a problem when $q_x = 0$, as H is not well defined there. Indeed, at that point motion of the pendulum does not result in any motion in the chosen y coordinates.

Instead, lets consider the change of coordinates to the angle of the pendulum relative to the $+x$ axis, $y = \begin{bmatrix} y_\theta \end{bmatrix}$, where $y_\theta = \psi(q) = \tan^{-1}(q_y/q_x)$. Thus, we have,

$$Y = D\psi = \begin{bmatrix} \frac{-q_y}{q_x^2+q_y^2} & \frac{q_x}{q_x^2+q_y^2} \end{bmatrix} \tag{42}$$

$$H = \begin{bmatrix} A \\ Y \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2q_x & 2q_y \\ \frac{-q_y}{q_x^2+q_y^2} & \frac{q_x}{q_x^2+q_y^2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{43}$$

$$= \frac{1}{\frac{2q_x^2}{q_x^2+q_y^2} - \frac{-2q_y^2}{q_x^2+q_y^2}} \begin{bmatrix} \frac{q_x}{q_x^2+q_y^2} & -2q_y \\ \frac{q_y}{q_x^2+q_y^2} & 2q_x \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -q_y \\ q_x \end{bmatrix} = \begin{bmatrix} -l\sin(y_\theta) \\ l\cos(y_\theta) \end{bmatrix} \tag{44}$$

To check our intuition, recall that $q_x = l\cos(y_\theta)$ and $q_y = l\sin(y_\theta)$. Thus, from $\dot{q} = H\dot{y}$, we see that $\dot{q}_x = -l\sin(y_\theta)\dot{y}_\theta$ and $\dot{q}_y = l\cos(y_\theta)\dot{y}_\theta$, which we can confirm geometrically. Checking the properties, we see that $AH = 0$ and $YH = 1$.

This change of coordinates is valid everywhere, as we need $q_x^2 + q_y^2 \neq 0$, i.e. we are not at the origin and $l \neq 0$.

3 Dynamics

To use our change of coordinates for dynamics, we can start by substituting in our definitions of $\dot{q}$ and $\ddot{q}$ in terms of $\dot{y}$ and $\ddot{y}$, (8) and (9) in general or (27) and (33) for the reduced coordinate case (which are equal but with different definitions of H). Then, by premultiplying the equations of motion by H^T, we get,

$$M\ddot{q} + C\dot{q} + N + A^T\lambda = \Upsilon \tag{45}$$

$$H^T M H\ddot{y} + H^T M\dot{H}\dot{y} + H^T C H\dot{y} + H^T N + H^T A^T \lambda = H^T \Upsilon \tag{46}$$

Collecting these terms, and noting in the general case that $AH = \tilde{A}$, (13),

$$\boxed{\tilde{M}\ddot{y} + \tilde{C}\dot{y} + \tilde{N} + \tilde{A}^T\lambda = \tilde{\Upsilon}} \tag{47}$$

or, in the reduced coordinate case, $AH = 0$ (35) and the λ term drops out,

$$\boxed{\tilde{M}\ddot{y} + \tilde{C}\dot{y} + \tilde{N} = \tilde{\Upsilon}} \tag{48}$$

where,

$$\tilde{M} = H^T M H \qquad \tilde{C} = H^T M\dot{H} + H^T C H \qquad \tilde{N} = H^T N \qquad \tilde{\Upsilon} = H^T \Upsilon \tag{49}$$

If our original dynamics were constrained but $\dot{y}$ is not, we can now work with these unconstrained, reduced dynamics to solve for $\ddot{y}$ and $\dot{y}$. Then we can reconstruct $\ddot{q}$ and $\dot{q}$ using (27) and (33).

3.1 Example Continued

Continuing the pendulum example from Section 2.3, lets compare the reduced dynamics derivation with a direct computation of the unconstrained dynamics. Assume the pendulum has a point mass m at the end of the length l pendulum. The dynamics in terms of $q = \begin{bmatrix} q_x & q_y \end{bmatrix}^T$ are (using A from (38)),

$$M\ddot{q} + C\dot{q} + N + A^T\lambda = \Upsilon \tag{50}$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{q}_x \\ \ddot{q}_y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_x \\ \dot{q}_y \end{bmatrix} + \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} 2q_x \\ 2q_y \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{51}$$

Instead, if we compute the dynamics in terms of $y = \begin{bmatrix} y_\theta \end{bmatrix}$, we get,

$$\begin{bmatrix} ml^2 \end{bmatrix} \begin{bmatrix} \ddot{y}_\theta \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} \dot{y}_\theta \end{bmatrix} + \begin{bmatrix} -mgl\cos(y_\theta) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{52}$$

Comparing this to the equations of motion derived from our reduced dynamics formula and H from (44),

$$\tilde{M} = \begin{bmatrix} -l\sin(y_\theta) & l\cos(y_\theta) \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} -l\sin(y_\theta) \\ l\cos(y_\theta) \end{bmatrix} = \begin{bmatrix} ml^2(\sin^2(y_\theta) + \cos^2(y_\theta)) \end{bmatrix} = \begin{bmatrix} ml^2 \end{bmatrix} \tag{53}$$

$$\tilde{C} = \begin{bmatrix} -l\sin(y_\theta) & l\cos(y_\theta) \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} -l\cos(y_\theta) \\ -l\sin(y_\theta) \end{bmatrix} \dot{y}_\theta + \begin{bmatrix} -l\sin(y_\theta) & l\cos(y_\theta) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -l\sin(y_\theta) \\ l\cos(y_\theta) \end{bmatrix} \tag{54}$$

$$= \begin{bmatrix} ml^2(\sin(y_\theta)\cos(y_\theta) - \cos(y_\theta)\sin(y_\theta)) \end{bmatrix} \dot{y}_\theta + \begin{bmatrix} 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{55}$$

$$\tilde{N} = \begin{bmatrix} -l\sin(y_\theta) & l\cos(y_\theta) \end{bmatrix} \begin{bmatrix} 0 \\ -mg \end{bmatrix} = \begin{bmatrix} -mgl\cos(y_\theta) \end{bmatrix} \tag{56}$$

$$\tilde{\Upsilon} = \begin{bmatrix} -l\sin(y_\theta) & l\cos(y_\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{57}$$

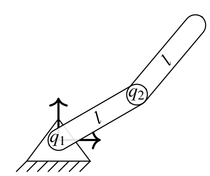
We see that the reduced dynamics compute the correct answer,

$$\tilde{M}\ddot{y} + \tilde{C}\dot{y} + \tilde{N} = \tilde{\Upsilon} \tag{58}$$

$$\begin{bmatrix} ml^2 \end{bmatrix} \begin{bmatrix} \ddot{y}_\theta \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} \dot{y}_\theta \end{bmatrix} + \begin{bmatrix} -mgl\cos(y_\theta) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{59}$$

4 Practice Problems

1) Calculate the Cartesian workspace dynamics for a two link manipulator in the plane. Derive the dynamics in terms of the configuration space $q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$. Then, use that to calculate the dynamics in the workspace coordinates of $y = \begin{bmatrix} y_x & y_y \end{bmatrix}^T$. Assume each link has length l, mass m, and uniform mass distribution along its length.



2) Derive the identity in (34), that,

$$\dot{H} = \begin{bmatrix} A \\ Y \end{bmatrix}^{-1} \begin{bmatrix} -\dot{A} \\ -\dot{Y} \end{bmatrix} H$$

3) Compute the dynamics for a dynamic unicycle model. Consider a single wheel in the horizontal plane. The wheel has a constraint that it cannot move sideways. Use $q = \begin{bmatrix} q_x & q_y & q_\phi \end{bmatrix}^T$ as the position and orientation of the wheel. Then, convert the velocity to the reduced coordinates $\dot{y} = \begin{bmatrix} \dot{y}_v & \dot{y}_\omega \end{bmatrix}^T$, representing the forward velocity and rotational velocity. The control inputs are f in the forward direction and τ in the rotational direction. First, find the constrained dynamics of the $\ddot{q}$ system. Then, use that to find the reduced dynamics of the $\ddot{y}$ system.

References

[1] A. M. Johnson and D. E. Koditschek, “Legged self-manipulation,” *IEEE Access*, vol. 1, pp. 310–334, May 2013.