Optimal Multi-Dimensional Mechanisms: A Heuristic Approach

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Introduction

- An old problem: how to design mechanisms with transfers
  - Sell multiple goods
  - Choose different dimensions of quality
- Type space: multiple dimensions
- The goal: come up with a method that is somewhat tractable and applicable to a range of problems
- Today: Focus on selling two goods

Literature

- Early papers: Palfrey (1983), McAfee and McMillan (1988), Armstrong (1996, 1999), Rochet and Chone (1998), Rochet and Stole (2003)
- Monopolistic screening: Manelli and Vincent (2006, 2007), Pavlov (2011ab), Chu et al. (2011), Hart and Reny (2015ab), Daskalakis et al. (2017), Hart and Nissan (2017), Kleiner and Manelli (2019), Armstrong and Vickers (2019), Haghpanah and Hartline (2020), Bikhchandani and Mishra (2021),
- Cyclical monotonicity: Rochet (1987), Kushnir and Lokutsievskiy (2021)
- Other papers/approaches: Goeree and Kushnir (2011), Kleiner et al. (2021)

The Model

- Seller selling two goods:

$$p$$

- A buyer

$$\sum_{i=1}^{2} x_i q_i - p = x \cdot q - p$$

The Model

- buyer's type, x, is distributed according to density $f(x)$ with:
  - Support is $[0,1]^2 = X$
  - density $f(x)$ is C^1
- Our results work when there is a cost for q: e.g., $c(q) = \sum_{i=1}^{2} q_i^2/2$

Mechanisms

- Mechanism: $\{p(x), q(x)\}_{x \in X}; p : X \to \mathbb{R}, q : X \to \mathbb{R}^2_+$
- IC: $x \cdot q(x) - p(x) \geq x \cdot q(x') - p(x'), \forall x', x$
- IR: $x \cdot q(x) - p(x) \geq 0$

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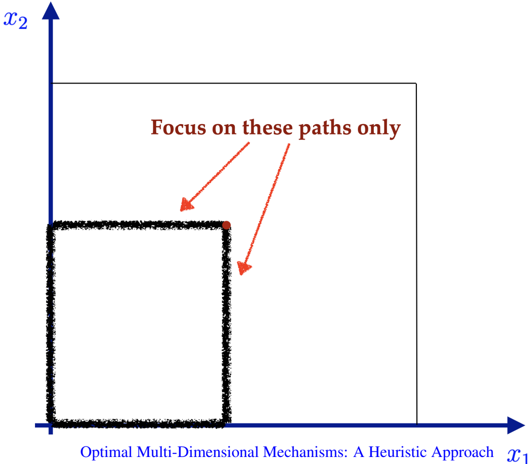
Proposition. Rochet (1987). An allocation is incentive compatibility if and only if there exists a convex potential function $u(x)$ such that $\nabla u(x) = q(x)$.

The Main Idea: Heuristics

- For every two paths between x and 0, there should be a constraint: $\int q \cdot dr_1 = \int q \cdot dr_2$
  - Too many constraints!!!
- The main heuristic:



The Main Idea: More Formal

- Formally, we show a semi/strong duality result that can handle any finite class of paths
- Use it to look at class of problems where considering two classes of paths are enough
- Hope that a potential function exists
- Reason for hope(!):

The Main Idea: More Formal

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- Reason for hope(!):

Green (Stokes) Theorem. If $q : X \to \mathbb{R}^2$ is continuously differentiable, then $\frac{\partial q_1}{\partial x_2}(x) = \frac{\partial q_2}{\partial x_1}(x), \forall x \in X$ implies $\oint_C q \cdot dx = 0$ for all piece-wise smooth closed loops in X.

Mechanism Design Problem

$$\sup_{p:X\to\mathbb{R},q:X\to[0,1]^2,u:X\to\mathbb{R}_+} \int p(x) f(x)\, dx$$

subject to

$$u(x) : \text{convex}$$
$$u(x) = x \cdot q(x) - p(x)$$
$$\nabla u(x) = q(x)$$

Mechanism Design Problem

$$\sup_{q:X\to[0,1]^2, u:X\to\mathbb{R}_+} \int [x\cdot\nabla u(x) - u(x)] f(x)\, dx$$

subject to

$$u(x) : \text{convex}$$

$$\cancel{u(x) = x \cdot q(x) - p(x)}$$

$$\nabla u(x) = q(x)$$

Mechanism Design Problem

$$\sup_{q:X\to[0,1]^2,\, u:X\to\mathbb{R}_+} \int u(x)\, d\mu$$

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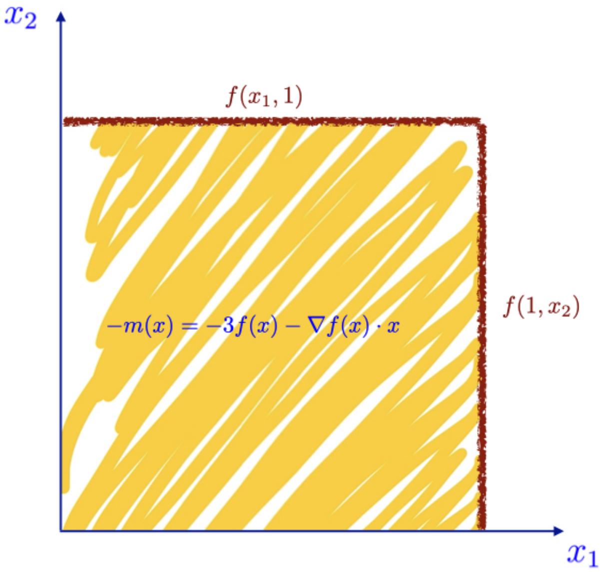
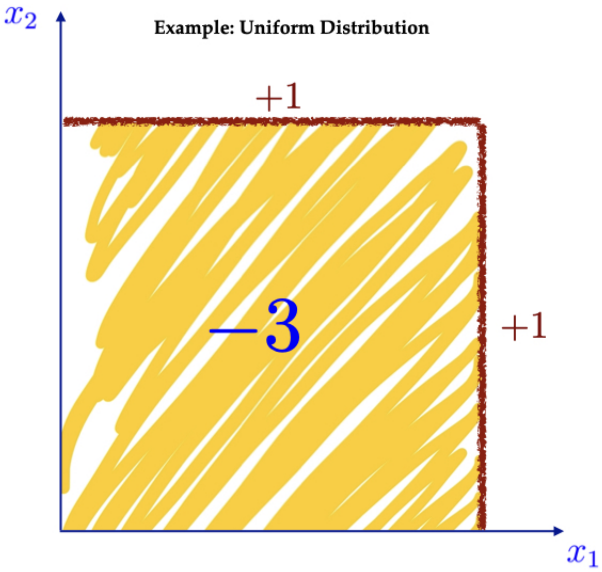
Use Green-Stokes Theorem

$$\mu(A) = \int_A -(3f(x) + \nabla f(x)\cdot x)\, dx$$
$$+ \int_{A\cap[0,1]\times\{1\}} f(x_1, 1)\, dx_1 + \int_{A\cap\{1\}\times[0,1]} f(1, x_2)\, dx_2$$

The signed measure μ



The signed measure μ



Mechanism Design Problem – Relaxed

$$\sup_{u:X\to\mathbb{R}_+} \int u(x)\,d\mu$$

subject to

$$u \in \left\{ u | \exists q : X \to [0,1]^2, u(x) = \int_0^{x_1} q_1(t,0)\,dt + \int_0^{x_2} q_2(x_1,t)\,dt \right\} = F_1$$

$$u \in \left\{ u | \exists q : X \to [0,1]^2, u(x) = \int_0^{x_2} q_2(0,t)\,dt + \int_0^{x_1} q_1(t,x_2)\,dt \right\} = F_2$$

Basic idea: we can use Lagrange multipliers associated with the equality constraint. Need a strong duality result to be able to use them

Mechanism Design Problem – Relaxed

$$\sup_{u \in F_1 \cap F_2} \langle u, \mu \rangle$$

We can view the above as a function of μ over the convex set $F_1 \cap F_2$.

$$\mathscr{S}(\omega | C) = \sup_{u \in C} \langle u, \omega \rangle$$

C is convex subset of a t.v.s. $\mathcal{F}$; $\omega \in \mathcal{F}^*$ a member of the dual of $\mathcal{F}^*$.

Support Functional

- $\mathscr{S}(\cdot|\cdot)$ is the support functional for set C in the direction ω – see Kushnir (several!)
- What is the space $\mathcal{F}$?

$$\mathcal{F} = L_2(X) \times L_2([0,1] \times \{1\}) \times L_2(\{1\} \times [0,1])$$
$$\mathcal{F}^* = \mathcal{F}$$

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$$\mathcal{F}^* = \mathcal{F}$$

- Recall that

$$\mu(A) = \int_A -(3f(x) + \nabla f(x) \cdot x)\, dx + \int_{A \cap [0,1] \times \{1\}} f(x_1, 1)\, dx_1 + \int_{A \cap \{1\} \times [0,1]} f(1, x_2)\, dx_2$$

So $\mu \in \mathcal{F}^*$.

Strong Duality

Proposition. Support functional satisfies

$$\mathscr{S}(\mu|F_1 \cap F_2) = \inf_{\omega \in \mathcal{F}} \mathscr{S}(\omega|F_1) + \mathscr{S}(\mu - \omega|F_2) \qquad (\star)$$

- ω is the Lagrange multiplier associated with the equality constraint
- Key difficulty: typically need non-empty interior for F_1 and F_2 (Fenchel-Rockefellar Theorem)
- We follow Mitter (2008); see also Gretsky, Ostroy, and Zame (2002), Kleiner and Manelli (2019)
    - Strong duality is equivalent to non-empty subgradient of the value function
    - Suffiecient to show that when μ is shifted by $\delta\mu$, the objective changes by at least $M\left\|\delta\mu\right\|$ for some $M > 0$.

Strong Duality

Corollary. Suppose $q : X \to [0,1]^2$ and $\omega \in \mathcal{F}$ exists such that:

1. Relaxed IC holds:

$$u(x) = \int_0^{x_1} q_1(t,0)\,dt + \int_0^{x_2} q_2(x_1,t)\,dt$$
$$u(x) = \int_0^{x_2} q_2(0,t)\,dt + \int_0^{x_1} q_1(t,x_2)\,dt$$

2. We have

$$\mathscr{S}(\omega|F_1) = \langle u, \omega\rangle, \mathscr{S}(\mu-\omega|F_2) = \langle u, \mu-\omega\rangle$$

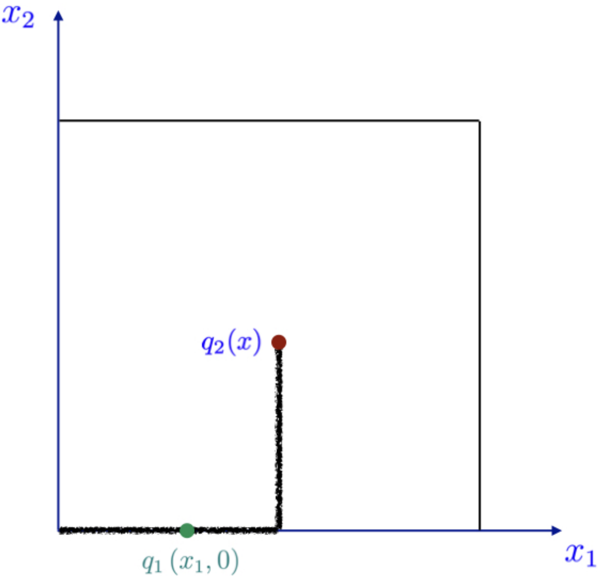
Then, $\mathscr{S}(\mu|F_1 \cap F_2) = \langle u, \mu\rangle$.

From 2D to 1D

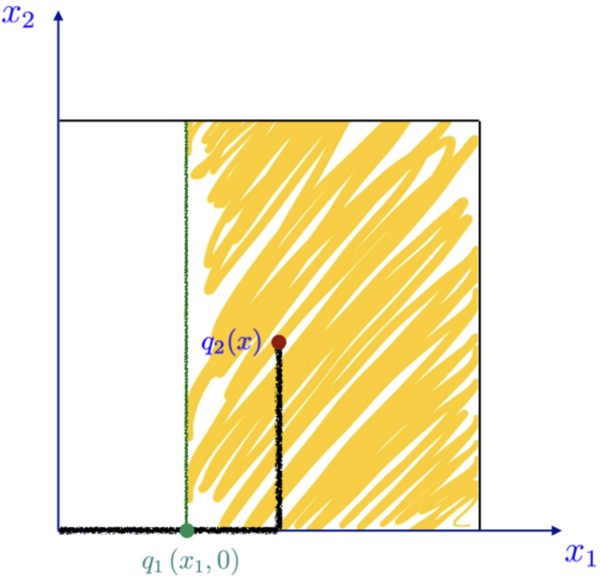
- How do we find u such that $\mathscr{S}(\omega|F_1) = \langle u, \omega \rangle$?
- This is identical to the 1D problem, e.g. Myerson, Mussa-Rosen, etc.
- Pointwise optimization:

$$\langle u, \omega \rangle = \int \left[\int_0^{x_1} q_1(t, 0)\, dt + \int_0^{x_2} q_2(x_1, t)\, dt \right] d\omega$$

From 2D to 1D



From 2D to 1D



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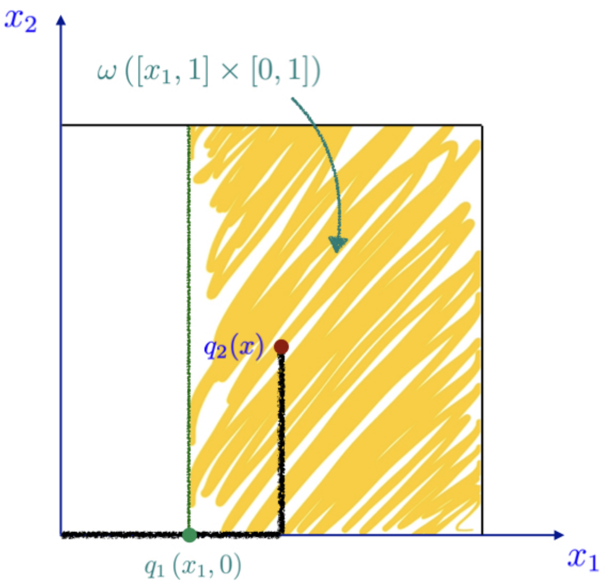

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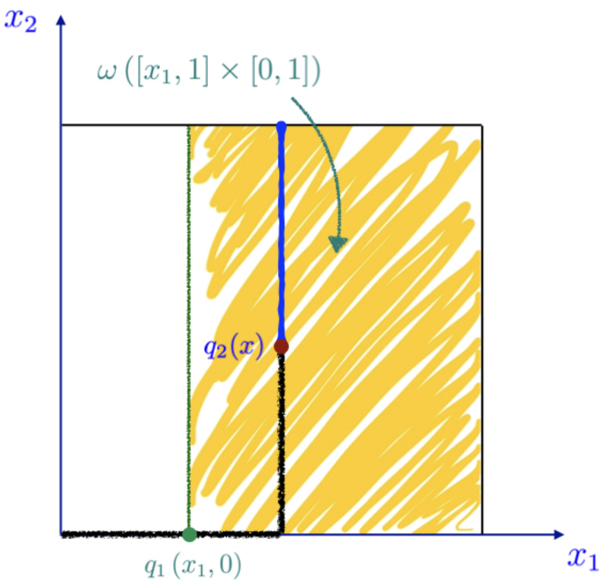

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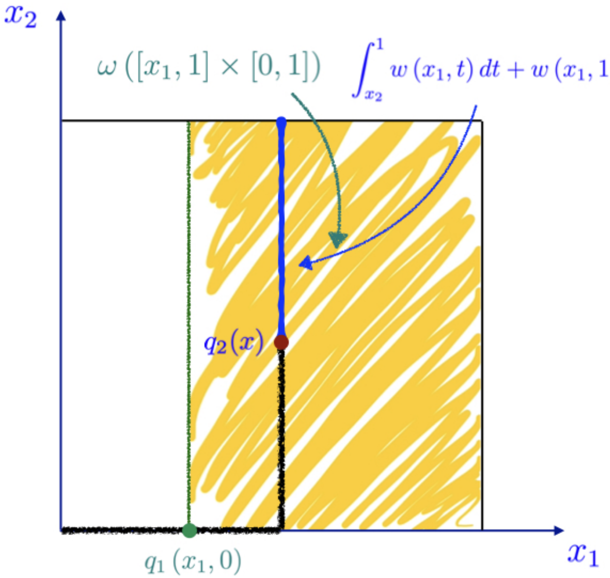
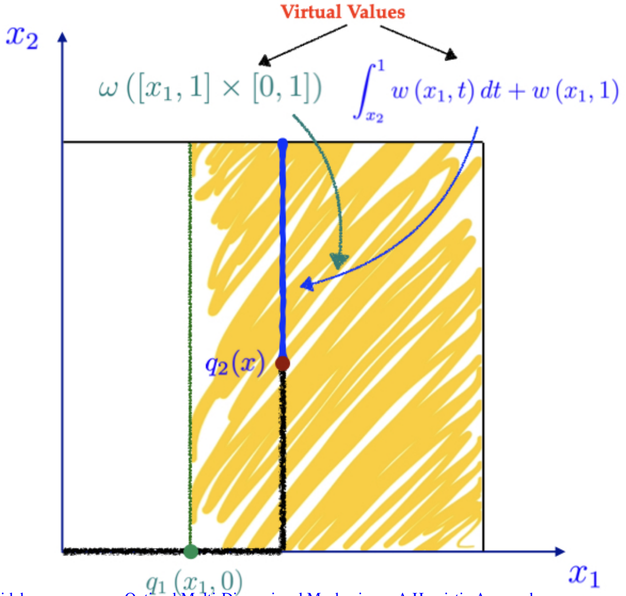

x_2
$\omega\left([x_1, 1] \times [0, 1]\right)$
$\int_{x_2}^{1} w\left(x_1, t\right) dt + w\left(x_1, 1\right)$
$q_2(x)$
$q_1\left(x_1, 0\right)$
x_1

From 2D to 1D



Virtual Values
x_2
$\omega([x_1, 1] \times [0, 1])$
$\int_{x_2}^{1} w(x_1, t)\, dt + w(x_1, 1)$
$q_2(x)$
$q_1(x_1, 0)$
x_1

Optimal 1D Allocations

- Optimal allocations for F_1:

$$q_1(x_1,0) = \begin{cases} 1 & \omega([x_1,1]\times[0,1]) > 0 \\ \in [0,1] & \omega([x_1,1]\times[0,1]) = 0 \\ 0 & \omega([x_1,1]\times[0,1]) < 0 \end{cases}$$

$$q_2(x) = \begin{cases} 1 & \int_{x_2}^{1} w(x_1,t)\,dt + w(x_1,1) > 0 \\ \in [0,1] & \int_{x_2}^{1} w(x_1,t)\,dt + w(x_1,1) = 0 \\ 0 & \int_{x_2}^{1} w(x_1,t)\,dt + w(x_1,1) < 0 \end{cases}$$

- Optimal allocations for F_2 is similar

Symmetry

Assumption. Density $f(x)$ is symmetric, i.e., $f(x) = f\left(x^T\right)$.

- When f is symmetric, so is μ. We can conjecture that

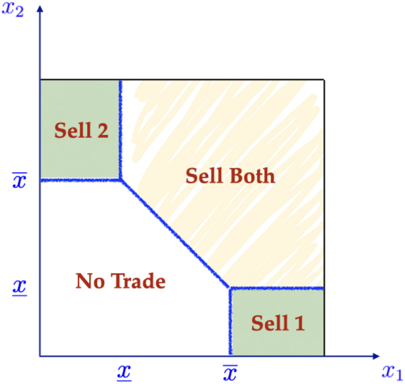
$$\omega(A) = \int_A w(x)\,dx + \int_{A\cap[0,1]\times\{1\}} f(x_1, 1)\,dx_1$$

$$(\mu - \omega)(A) = \int_A \left(-m(x) - w\left(x^T\right)\right) dx + \int_{A\cap\{1\}\times[0,1]} f(1, x_2)\,dx_2$$

- So if we can come up with $w(x) = -m(x) - w\left(x^T\right)$ we are done!

Deterministic Mechanisms

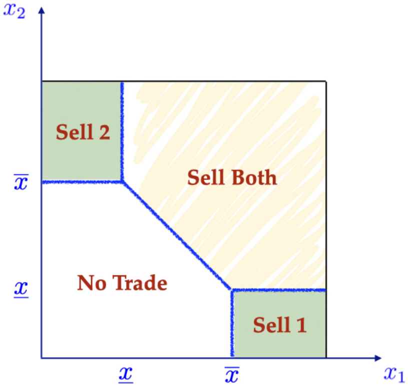
- 1D: (revneue maximizing) optimal mechanism is determinic – posted price: Myerson (1981), Riley and Zeckhauser (1983)
- 2D: when is this the case?
- A symmetric deterministic mechanism should have this form:



Deterministic Mechanisms

Proposition. If the solution to (★) is symmetric and deterministic, then

$$-\int_{\overline{x}}^{1} m(x_1,t)\,dt + f(x_1,1) = 0, \forall x_1 \in [0,\underline{x}]$$



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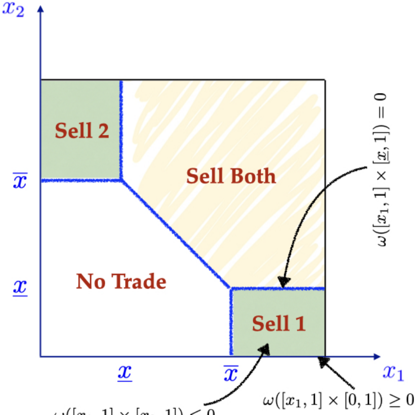

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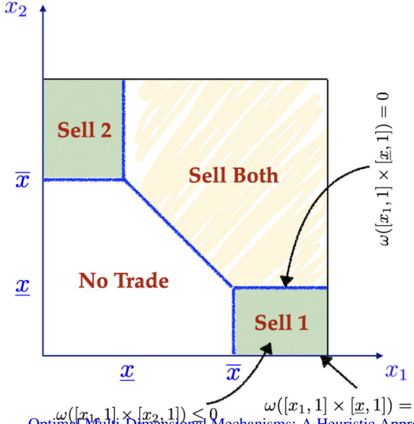

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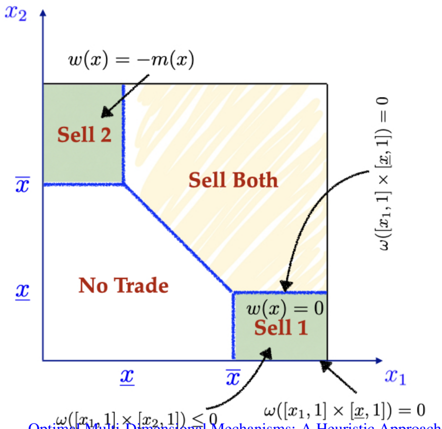


Deterministic Mechanisms

Corollary. If $f(x) = g(x_1)\,g(x_2)$ and the solution to ($\bigstar$) is symmetric and deterministic, then:

1. g has power form

$$g(x) = (1+\alpha)\,x^{\alpha},$$

2. Upper cutoff satisfies

$$\bar{x} = \left(\frac{2+\alpha}{2\alpha+3}\right)^{\frac{1}{1+\alpha}}$$

   Note: Monopoly posted price is $\left(\frac{1}{2+\alpha}\right)^{\frac{1}{1+\alpha}} < \bar{x}$.

3. The lower cutoff satisfies

$$\int_{\underline{x}}^{1}\left[f(x_1,1) - \frac{1}{2}\int_{\max\{\underline{x}+\bar{x}-x_1,x_1\}}^{1} m(x_1,t)\,dt\right]dx_1 = 0$$

Integral Equation

- Finding virtual values comes down to solving integral equations:

$$w(x) + w\left(x^T\right) = -m(x), \forall x$$

$$f(x_1, 1) + \int_{\max\{\underline{x}+\bar{x}-x_1, \underline{x}\}}^{1} w(x_1, t)\, dt = 0$$

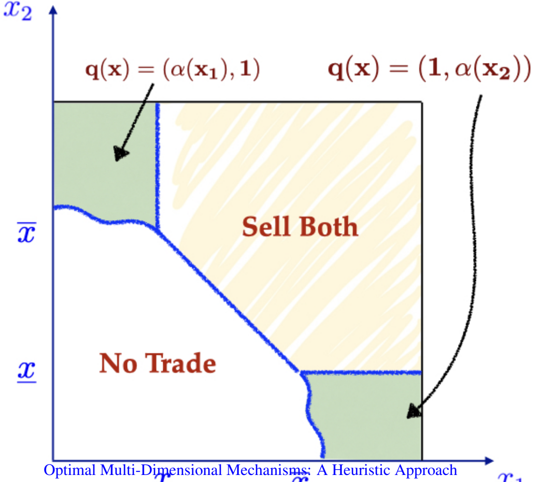
- We can construct several solutions. Existence is equivalent to virtual values being positive over the "both sell" region. This is somewhat easy to check.

Deterministic Mechanisms

- Calculations suggest that for $\alpha \geq 0$, $f(x) = (1+\alpha)^2 (x_1 x_2)^\alpha$ virtual values are indeed positive in the both-sell region.
  - Still needs to be proved!

A (somewhat) Generic Class of Random Mechanisms

- Losely speaking, the restriction for deterministic mechanisms is fairly non-generic.
- Alternative mechanism:



Random Mechanisms

- The both-sell region is identical to deterministic.
- Integral equation for virtual values is the same
- We are still looking for sufficient conditions for positive virtual values

Auctions

- The strong duality result holds for an number of buyers
- Optimal: Allocate each good to to the buyer with highest positive virtual value

Proposition. With any number of buyers the participation regions are the same.

- Integral equation is much more difficult to solve.
  - IC depends on the level curves of virtual values
- Unlike 1D, optimal allocations depend on the number of buyers:
  - interim quantities $Q_n(x)$ are CDF's of highest positive virtual value for good n
  - Incentive compatbility

$$\frac{\partial Q_1(x)}{\partial x_2} = \frac{\partial Q_2(x)}{\partial x_1}$$

Conclusion

- Developed a heuristic method to solve 2D mechanisms design problems with transfers
- Finding optimal allocations often equivalent to solving integral equations
- Very much work in progress.