Euclid's *Elements* and Diagrammatic Reasoning in Geometry

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(joint work with Ed Dean and John Mumma)

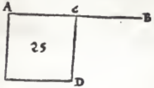
The *Elements*

For more than two thousand years, Euclid's *Elements* was held to be the paradigm for rigorous argumentation.

exemplum, si quis dicat, diuide 10 in duas partes, ex quarum unius in reliquam ductu, producatur 30, aut 40, manifestum est, quòd casus seu quæstio est imposibilis, sic tamē operabimur, diuidemus 10 per æqualia, & fiet eius medietas 5, duc in se fit 25, auferes ex 25, ipsum producendum, utpote 40, ut docui te, in capitulo operationum, in sexto libro, fiet residuum m: 15, cuius ℞ addita & detracta a 5, ostendit partes, quæ inuicem ductæ producunt 40, erunt igitur hæ, 5 p: ℞ m: 15, & 5 m: ℞ m: 15.

DEMONSTRATIO

Vt igitur regulæ uerus pateat intellectus, sit A B linea, quę dicatur 2
10, diuidenda in duas partes, quarū rectangulum debeat esse 40, est aūt 40 q̄druplū ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, absq; numero, ℞ igitur residui, si aliquid maneret, addita & detracta ex A C, ostenderet partes, at quia tale residuum est minus, ideo imaginaberis ℞ m: 15, id est differentiæ A D, & quadrupli A B, quam adde & minue ex A C, & habebis quæsitum, scilicet 5 p: ℞ v: 25 m: 40, & 5 m: ℞ v: 25 m: 40, seu 5 p: ℞ m: 15, & 5 m: ℞ m: 15, duc 5 p: ℞ m: 15 in 5 m: ℞ m: 15, dimissis incruciationibus, fit 25 m: m: 15, quod est p: 15, igitur hoc productum est 40, natura tamē A D, non est eadem cū natura 40, nec A B, quia superficies est remota à natura numeri, & lineæ, proximius tamē huic quantitati, quę uere est sophistica, quoniam per eam, non ut in puro m: nec in alijs, operationes exercere licet, nec uenari quid sit est, ut addas quadratum medietatis numeri numero producendo, & à ℞ aggregati minuas ac addas dimidium diuidendi. Exem



5 p: ℞ m: 15
5 m: ℞ m: 15
25 m: m: 15 q̄d. est 40

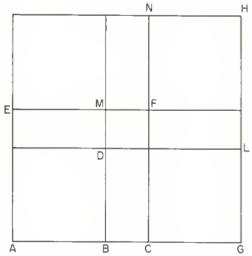


Fig. 1

If therefore AD [Fig. 1] is made 1 fourth power[1] and CD and DE are made 3 squares, and DF is made 9, BA will necessarily be a square and BC will necessarily be 3. Since we wish to add some squares to DC and DE, let these [additions] be [the rectangles] CL and KM. Then in order to complete the square it will be necessary to add the area LNM. This has been shown to consist of the square on GC, which is half the number of [added] squares, since CL is the area [made] from [the product of] GC times AB, where AB is a square, AD having been assumed to be a fourth power. But FL and MN are each equal to GC times CB, by Euclid I, 42,[2] and hence the area LMN, which is the number to be added, is a sum composed of the product of GC into twice CB, that is, into the number of squares which was 6, and GC into itself, which is the number of squares to be added. This is our proof [of the possibility of a solution].

This having been completed, you will always reduce the part containing the fourth power to a root, viz, by adding enough to each side so that the fourth power with the square and number may have a root. This is easy when you take half the number of the squares as the root of the number; and you will at the

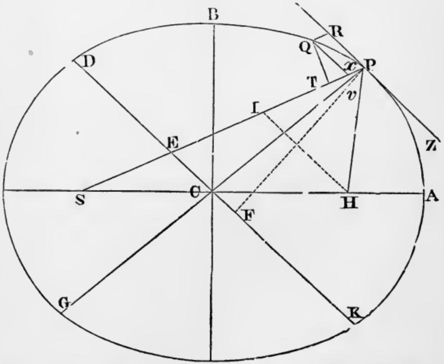
[1] Cardan writes "square-square," quadratum quadratum ($q^d q^d$), hence x^4.

SECTION III.

Of the motion of bodies in eccentric conic sections.

PROPOSITION XI. PROBLEM VI.

If a body revolves in an ellipsis; it is required to find the law of the centripetal force tending to the focus of the ellipsis.



Let S be the focus of the ellipsis. Draw SP cutting the diameter DK of the ellipsis in E, and the ordinate Qv in x; and complete the parallelogram QxPR. It is evident that EP is equal to the greater semi-axis AC: for drawing HI from the other focus H of the ellipsis parallel to EC, because CS, CH are equal, ES, EI will be also equal; so that EP is the half sum of PS, PI, that is (because of the parallels HI, PR, and the equal angles IPR, HPZ), of PS, PH, which taken together are equal to the whole axis 2AC. Draw QT perpendicular to SP, and putting L for the princi al latus rectum of the ellipsis (or for

EXPLICATIO.

Dico abſolutè infinitum, non autem in ſuo genere; quicquid enim in ſuo genere tantùm infinitum eſt, infinita de eo attributa negare poſſumus; quod autem abſolutè infinitum eſt, ad ejus eſſentiam pertinet, quicquid eſſentiam exprimit, & negationem nullam involvit.

VII. Ea res libera dicetur, quæ ex ſolâ ſuæ naturæ neceſſitate exiſtit, & à ſe ſolâ ad agendum determinatur: Neceſſaria autem, vel potiùs coacta, quæ ab alio determinatur ad exiſtendum, & operandum certâ, ac determinatâ ratione.

VIII. Per æternitatem intelligo ipſam exiſtentiam, quatenus ex ſolâ rei æternæ definitione neceſſariò ſequi concipitur.

EXPLICATIO.

Talis enim exiſtentia, ut æterna veritas, ſicut rei eſſentia, concipitur, proptereaque per durationem, aut tempus explicari non poteſt, tametſi duratio principio, & fine carere concipiatur.

AXIOMATA.

I. Omnia, quæ ſunt, vel in ſe, vel in alio ſunt.

II. Id, quod per aliud non poteſt concipi, per ſe concipi debet.

III. Ex datâ cauſâ determinatâ neceſſariò ſequitur effectus, & contrà, ſi nulla detur determinata cauſa, impoſſibile eſt, ut effectus ſequatur.

IV. Effectûs cognitio à cognitione cauſæ dependet, & eandem involvit.

V. Quæ nihil commune cum ſe invicem habent, etiam per ſe invicem intelligi non poſſunt, ſive conceptus unius alterius conceptum non involvit.

VI. Idea

VI. Idea vera debet cum ſuo ideato convenire.

VII. Quicquid, ut non exiſtens, poteſt concipi, ejus eſſentia non involvit exiſtentiam.

PROPOSITIO I.

Subſtantia prior eſt naturâ ſuis affectionibus.

DEMONSTRATIO

Patet ex *Definitione* 3 & 5.

PROPOSITIO II.

Duæ ſubſtantiæ, diverſa attributa habentes, nihil inter ſe commune habent.

DEMONSTRATIO

Patet etiam ex *Defin.* 3. Unaquæque enim in ſe debet eſſe, & per ſe debet concipi, ſive conceptus unius conceptum alterius non involvit.

PROPOSITIO III.

Quæ res nihil commune inter ſe habent, earum una alterius cauſa eſſe non poteſt.

DEMONSTRATIO.

Si nihil commune cum ſe invicem habent, ergo *(per Axiom.* 5.) nec per ſe invicem poſſunt intelligi, adeóque *(per Axiom.* 4.) una alterius cauſa eſſe non poteſt. *Q. E. D.*

PROPOSITIO IV.

Duæ, aut plures res diſtinctæ, vel inter ſe diſtinguuntur, ex diverſitate attributorum ſubſtantiarum, vel ex diverſitate earundem affectionum.

DEMONSTRATIO.

Omnia, quæ ſunt, vel in ſe, vel in alio ſunt, *(per Axiom.* 1.) hoc eſt *(per Defin.* 3. & 5.) extra intellectum nihil datur præter ſubſtantias, ejuſque affectiones. Nihil ergo extra intellectum datur, per quod plures res diſtingui inter ſe poſſunt præter ſubſtantias, ſive quod

 idem

The *Elements*

The nineteenth century raised concerns:

- Conclusions are drawn from diagrams, using "intuition" rather than precise rules.
- Particular diagrams are used to infer general results (without suitable justification).

Axiomatizations due to Pasch and Hilbert, and Tarski's formal axiomatization later on, were thought to make Euclid rigorous.

The *Elements*

But in some ways, they are unsatisfactory.

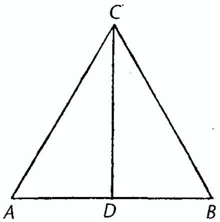
- Proofs in the new systems look very different from Euclid's.
- The initial criticisms belie the fact that Euclidean practice was remarkably stable for more than two thousand years.

Our project (Mumma, Dean, and me):

- Describe a formal system that is much more faithful to Euclid.
- Argue that the system is sound and complete (for the theorems it can express) relative to Euclidean fields.
- Show that the system can easily be implemented using contemporary automated reasoning technology.

Proposition 10

To bisect a given finite straight line.



Let *AB* be the given finite straight line.

Thus it is required to bisect the finite straight line *AB*.

Let the equilateral triangle *ABC* be constructed on it, [I. 1]
and let the angle *ACB* be bisected by the straight line *CD*; [I. 9]

I say that the straight line *AB* has been bisected at the point *D*.

For, since *AC* is equal to *CB*, and *CD* is common,
the two sides *AC*, *CD* are equal to the two sides *BC*, *CD* respectively;
and the angle *ACD* is equal to the angle *BCD*;
therefore the base *AD* is equal to the base *BD*. [I. 4]

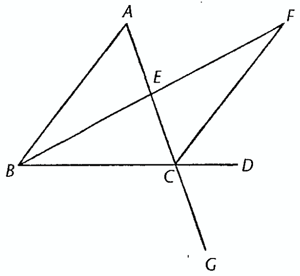
Therefore the given finite straight line *AB* has been bisected at *D*.

Q.E.F.

Proposition 16

In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*;
I say that the exterior angle *ACD* is greater than either of the interior and opposite angles *CBA*, *BAC*.

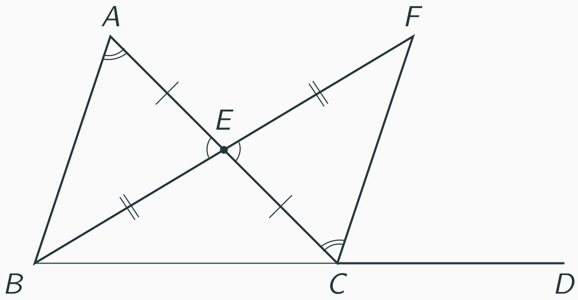


Let *AC* be bisected at *E*, **[I. 10]**
and let *BE* be joined and produced in a straight line to *F*;
let *EF* be made equal to *BE*, **[I. 3]**
let *FC* be joined, **[Post. 1]**
and let *AC* be drawn through to *G*. **[Post. 2]**

Then, since *AE* is equal to *EC*, and *BE* to *EF*,
the two sides *AE*, *EB* are equal to the two sides *CE*, *EF* respectively;
and the angle *AEB* is equal to the angle *FEC*, for they are vertical angles. **[I. 15]**

Therefore the base *AB* is equal to the base *FC*, and the triangle *ABE* is equal to

The nature of diagrammatic inference



By side-angle-side, $\triangle AEB \equiv \triangle CEF$. So $\angle BAC = \angle ACF$.

Clearly $\angle ACD > \angle ACF$. So $\angle ACD > \angle BAC$.

But why is it clear that $\angle ACD > \angle ACF$?

First salient feature: the use of diagrams

Observation: the diagram is inessential to the communication of the proof. (Rather, it is used to “see” that the inferences are correct.)

Exercise:

- Let p and q be points on a line.
- Let r be between p and q.
- Let s be between p and r.
- Let t be between r and q.

Is s necessarily between p and t?

Methodological stance: from a logical perspective, the way to characterize diagrammatic reasoning is in terms of the class of inferences that are licensed.

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First salient feature: the use of diagrams

Observation (Manders): In a Euclidean proof, diagrams are only used to infer "co-exact" (regional / topological) information, such as incidence, intersection, containment, etc.

Exact (metric) information, like congruence, is always made explicit in the text.

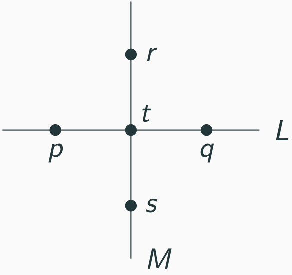
Poincaré: "Geometry is the art of precise reasoning from badly constructed diagrams."

Solution: take the "diagram" to be a representation of the relevant data.

Second salient feature: generality

Some aspects of diagrammatic inference are puzzling:

- Let p and q be distinct points.
- Let L be a line though p and q.
- Let r and s be on opposite sides of L.
- Let M be the line through r and s.
- Let t be the intersection of L and M.



Is t necessarily between r and s? Is t necessarily between p and q?

Not every feature found in a particular diagram is generally valid.

We need an explanation as to what secures the generality.

Third salient feature: logical form

Theorems in Euclid are of the form:

> *Given points, lines, circles, satisfying . . . , there are points, lines, circles satisfying . . .*

where each . . . is a conjunction of literals.

(If the inner existential quantifier is absent, it is a "demonstration" rather than a "construction.")

Proofs contain a construction part, and a deduction part.

Reasoning is linear, assertions are literals.

Exceptions: proof by contradiction, using a case distinction (sometimes "without loss of generality").

Fourth salient feature: nondegeneracy

In the statement of a theorem, points are generally assumed to be distinct, triangles are nondegenerate, etc.

Two issues:

- Sometimes the theorem still holds in some degenerate cases.
- When the theorems are applied, Euclid doesn't always check nondegeneracy.

I will have little to say about this; in our system, nondegeneracy requirement are stated explicitly.

Formalizing Euclid

Prior efforts:

- Nathaniel Miller's Ph.D. thesis (2001): system is very complicated; generality is attained by considering cases exhaustively.
- John Mumma's Ph.D. thesis (2006): employs diagrams (and equivalence relation on diagrams); generality is attained using rules.

Our formal system, E, is derived from Mumma's. But now a "diagram" is nothing more than an abstract representation of topological information. The system spells out what can be inferred from the diagram.

The language of E

Basic sorts:

- diagram sorts: points $p, q, r, \ldots$, lines $L, M, N, \ldots$, circles $\alpha, \beta, \gamma, \ldots$
- metric sorts: lengths, angles, and areas.

Basic symbols:

- diagram relations: on(p, L), same-side(p, q, L), between(p, q, r), on(p, γ), inside(p, γ), center(p, γ), intersects(L, M), $=$
- metric functions and relations: $+$, $<$, $=$, right-angle
- connecting functions: $\overline{pq}$, $\angle pqr$, $\triangle pqr$

Other relations can be defined from these; e.g.

$$\text{diff-side}(p, q, L) \equiv \neg\text{on}(p, L) \wedge \neg\text{on}(q, L) \wedge \neg\text{same-side}(p, q, L)$$

Sequents

The proof system establishes sequents of the following form:

$$\Gamma \Rightarrow \exists \vec{q}, \vec{M}, \vec{\beta}.\ \Delta$$

where Γ and Δ are sets of literals.

Applying a construction rule or prior theorem augments $\vec{q}$, $\vec{M}$, $\vec{\beta}$, Δ.

Applying deductive inferences augments Δ.

Case splits and suppositional reasoning temporarily augment Γ.

I need to describe:

- Construction rules.
- Deductive inferences.

Construction rules

"Let p be a point on L"

No prerequisites.

"Let p be a point distinct from q and r"

No prerequisites.

"Let L be the line through p and q"

Requires $p \neq q$.

"Let p be the intersection of L and M."

Requires that L and M intersect.

And so on...

Deductive inferences

Four types:

1. Diagram inferences: any fact that can be "read off" from the diagram.
2. Metric inferences: essentially linear arithmetic on lengths, angles, and areas.
3. Diagram to metric: for example, if q is between p and r, then $\overline{pq} + \overline{qr} = \overline{pr}$, and similarly for areas and angles.
4. Metric to diagram: for example, if p is the center of γ, q is on γ, and $\overline{pr} < \overline{pq}$, then r is inside γ.

Diagram inferences

Both construction inferences and diagram inferences require an account of what can be "read off" from the diagram.

We get this by closing the diagrammatic data in $\Gamma \cup \Delta$ under various rules, including:

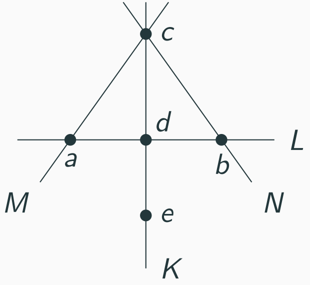
- properties of "between"
- properties of "same side"
- "Pasch rules," relating "between" and "same side"
- triple incidence rules
- circle rules
- intersection rules

These yield conclusions that are generally valid, that is, common to all possible realizations.

Proposition I.10. Assume a and b are distinct points on L.
Construct a point d such that d is between a and b, and $\overline{ad} = \overline{db}$.

By Proposition I.1 applied to a and b, let c be a point such that $\overline{ab} = \overline{bc}$ and $\overline{bc} = \overline{ca}$ and c is not on L.

Let M be the line through c and a.
Let N be the line through c and b.
By Proposition I.9 applied to a, c, b, M, N, let e be a point such that $\angle ace = \angle bce$, b and e are on the same side of M, and a and e are on the same side of N.
Let K be the line through c and e.
Let d be the intersection of K and L.
Hence $\angle ace = \angle acd$.
Hence $\angle bce = \angle bcd$.



By Proposition I.4 applied to a, c, d, b, c, d have $\overline{ad} = \overline{bd}$. Q.E.F.

Completeness

Tarski's first-order axiomatization of Euclidean geometry yields a complete theory of the Euclidean plane (inter-interpretable with real closed fields).

Drop the completeness axiom, and replace it with an axiom asserting that if a line L passes through a point inside a circle α, then L and α intersect.

The resulting theory is inter-interpretable with the theory of "Euclidean fields," and so is complete wrt "ruler and compass constructions." (Ziegler: it is also undecidable.)

Theorem. If a sequent of E is valid wrt to ruler and compass constructions, it can be derived in E.

Completeness

One strategy: interpret Tarski's theory in E.

Problem: Tarski includes full first-order logic!

Solution: With slight tinkering, Tarski's theory can be made "geometric," i.e. the axioms can be put in a restricted logical form.

A cut-elimination theorem due to Sara Negri then implies that any geometric assertion provable in Tarski's theory has a geometric proof.

Such a proof can be simulated in E.

Completeness

Outline of the proof:

1. Suppose a sequent A of E is valid for the intended semantics.
2. Then a translation $\pi(A)$ to Tarki's language is also valid for the intended semantics.
3. So it is provable in Tarski's theory.
4. So it has a cut-free proof.
5. This proof can be translated back to E, so E proves $\rho(\pi(A))$.
6. From this, E can derive the original sequent, A.

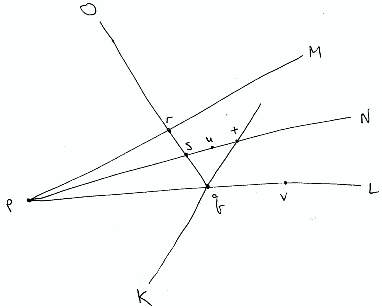
Implementation

Can we get a computer to carry out the diagrammatic inferences?

We experimented with:

- first-order theorem provers
- SMT solvers (CVC3, Z3)
- bespoke saturation procedures

SMT solvers were particularly good, and could carry out the metric inferences as well.



Data: all incidences (except on(u,N))

bet(p,s,t)

bet(q,s,r)

bet(s,u,t)

p ≠ q

¬ on(r,L)

```
:formula (sameside p t O)
:formula (sameside s t O)
:formula (not (sameside s t M))
:formula (not (sameside u t M))
:formula (bet s p t)
:formula (= M N)
:formula (bet q s u)
:formula (on q N)
:formula (= q t)
:formula (not (< (seg s u) (seg s t)))
:formula (not (< (seg u s) (seg s t)))
:formula (not (< (+ (seg s u) (seg u t)) (seg p t)))
:formula (not (< (+ (seg u s) (seg u t)) (seg p t)))
:formula (on u L)
:formula (on t L)
:formula (on p K)
:formula (not (sameside r s L))
:formula (not (sameside s u L))
:formula (not (sameside r u L))
:formula (sameside s v K)
:formula (not (= (+ (angle r p s) (angle s p q)) (angle r p q)))
:formula (not (sameside p s K))
:formula (not (sameside s t L))
:formula (= L K)
:formula (= q s)
:formula (= q t)
:formula (= q p)
:formula (not (= (+ (angle p q s) (angle s q t)) (angle p q t)))
:formula (not (< (angle p q s) (angle p q t)))
:formula (not (implies
  (= (+ (angle p q s) (angle s q t)) (angle p q t))
    (< (angle p q s) (angle p q t))))
```

Some conclusions

Our modest claims:

- We have a clean analysis of the type of reasoning that is used in books I–IV of the *Elements*.
- Our system is sound and complete for the expected semantics.
- The analysis makes it easy to verify formal texts that are very close to proofs in the *Elements*.
- This provides a clear sense in which the *Elements* is more rigorous than commonly acknowledged.
- We have analyzed the *logical form* of diagrammatic inference, separating these questions from cognitive, computational, pedagogical, and historical terms.
- The analysis can support further inquiry into *why* these inferences are basic to the practice.

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Counterexample Search in Diagram-Based Geometric Reasoning

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Abstract

Topological relations such as inside, outside, or intersection are ubiquitous to our spatial thinking. Here, we examined how people reason deductively with topological relations between points, lines, and circles in geometric diagrams. We hypothesized in particular that a counterexample search generally underlies this type of reasoning. We first verified that educated adults without specific math training were able to produce correct diagrammatic representations contained in the premisses of an inference. Our first experiment then revealed that subjects who correctly judged

2. Formal characterization of diagram-based geometric inferences

The formal characterization of diagram-based geometric inferences adopted in this study is based on the formal system developed by Avigad et al. (2009). Following this system, we considered a formal language $\mathscr{L}$ with three types of objects:

- *points* denoted by *A*, *B*, *C*, etc.,
- *lines* denoted by *L*, *M*, *N*, etc.,
- *circles* denoted by α, β, γ, etc.,

together with the following set of relations:

- point *A* is {inside, on, outside} circle α,
- point *A* is {on, off} line *L*,
- points *A* and *B* are {on the same side, on opposite sides} of line *L*,
- point *B* is {between, not between} points *A* and *C* on line *L*,
- line *L* {intersects, does not intersect} line *M*,
- line *L* {intersects, does not intersect} circle α,
- circle α {intersects, does not intersect} circle β,
- circle α is {inside, outside} circle β.

A proposition in the language $\mathscr{L}$ is always an atomic formula consisting of a single relation between particular geometric objects, for example, "point *A* is inside circle α" or "circle α intersects circle β."

In this setting a *diagram-based geometric inference* **I** is entirely characterized by a set of premisses and a conclusion in the language $\mathscr{L}$. Here are two examples of such inferences:

(I1)
Point *A* is *inside* circle α
Point *A* is *on* line *L*
____
Line *L* *intersects* circle α

(I2)
Point *A* is *outside* circle α
Point *A* is *on* line *L*
____
Line *L* *intersects* circle α

In order to say when a diagram-based geometric inference is valid or invalid, we have to define the notion of a model (in the logical sense) for a set of propositions in $\mathscr{L}$. A

4.1. Method

4.1.1. Design and materials

In this experiment, we presented 12 invalid scanning problems that were seen once with a diagram of low counterexample density and once with a diagram of high

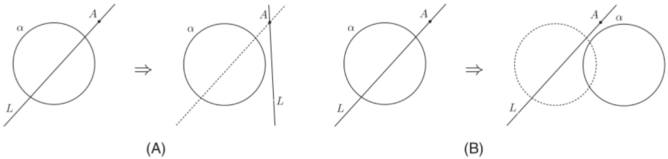

Fig. 5. Two examples of scanning operations starting with the diagram displayed in Fig. 2B. (A) Scanning operation with line *L*. (B) Scanning operation with circle α.

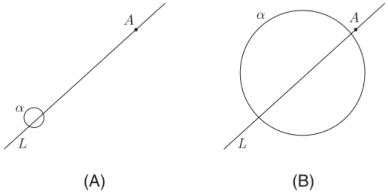

Fig. 6. Two possible diagrams in the case of inference I_2. When considering a scanning problem with line *L*, (A) is a diagram of *high* counterexample density and (B) is a diagram of *low* counterexample density.

ABSTRACT OF THE DISSERTATION

Formalized Synthetic Geometry

By ANDRÉ HERNÁNDEZ-ESPIET

Dissertation Director:

Alex Kontorovich

We offer a formalization of Book I of Euclid's Elements in Lean4 (with tools from Mathlib4) using System E by Avigad, Dean, and Mumma. We contrast the proofs in this system to those of Euclid's himself. We give detailed explanations for every theorem proved in Lean4 in order to complete Book I. Finally, we talk about the importance of tactics in Lean4 for the shortening and simplification of the proofs encountered in a geometric setting.

Autoformalizing Euclidean Geometry

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Abstract

Autoformalization involves automatically translating informal math into formal theorems and proofs that are machine-verifiable. Euclidean geometry provides an interesting and controllable domain for studying autoformalization. In this paper, we introduce a neuro-symbolic framework for autoformalizing Euclidean geometry, which combines domain knowledge, SMT solvers, and large language models (LLMs). One challenge in Euclidean geometry is that informal proofs rely on diagrams, leaving gaps in texts that are hard to formalize. To address this issue, we use theorem provers to fill in such diagrammatic information automatically, so that the LLM only needs to autoformalize the explicit textual steps, making it easier for the model. We also provide automatic semantic evaluation for autoformalized theorem statements. We construct LeanEuclid, an autoformalization benchmark consisting of problems

task, *autoformalization*: Can AI understand human-written problems/solutions and translate them automatically into formal theorems/proofs? Specifically, we focus on the setting where formal theorems/proofs can be verified by the Lean proof assistant (de Moura & Ullrich, 2021). Lean provides a language for writing formal proofs. It is popular among mathematicians and has a growing ecosystem of integration with large language models (LLMs), e.g., LeanDojo (Yang et al., 2023) and Lean Copilot (Song et al., 2024).

We demonstrate that Euclidean geometry provides an interesting and controllable domain for autoformalization. First, an automatic evaluation of autoformalized theorems is difficult in general but feasible in Euclidean geometry. Second, the logical gaps in informal proofs are well understood in Euclidean geometry, making it easier to faithfully formalize the proofs. Third, combining text-based and diagrammatic reasoning makes Euclidean geometry a natural domain to study multimodal reasoning models. Therefore, autoformalizing Euclidean geometry is an attractive target for AI.

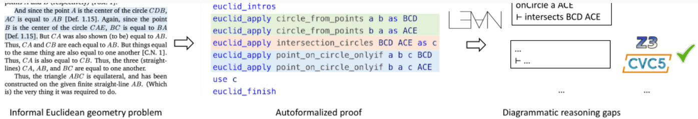

Figure 1. **Left**: Proposition 1 in Euclid's *Elements* (Book I). The orange text involves diagrammatic reasoning: Euclid did not explicitly prove the two circles actually intersect, but the reader can use the diagram to implicitly fill in the logical gap. **Top right**: The model autoformalizes the problem into a formal theorem (`proposition_1'`), which is evaluated by checking its logical equivalence with the ground truth (`proposition_1`), leveraging domain knowledge and a symbolic automated reasoning engine based on SMT (satisfiability modulo theories) solvers. **Bottom right**: A proof autoformalized by the model. Like Euclid's proofs, it does not need to handle diagrammatic reasoning explicitly. Lean can check the proof to identify a list of diagrammatic reasoning gaps, e.g., "`intersects BCD ACE`". Then, it attempts to fill in all gaps automatically using the symbolic reasoning engine based on SMT solvers.

bolic reasoning engine based on SMT solvers. As Fig. 1 (*Top right*) shows, given a ground-truth formal theorem T_{gt} and the autoformalized theorem T_{pred} produced by a language model, we use the symbolic engine to try to prove their equivalence ($T_{gt} \Leftrightarrow T_{pred}$). If successful, their logical gap is small enough to conclude that T_{pred} is correct. Even if the symbolic engine cannot prove $T_{gt} \Leftrightarrow T_{pred}$, it can provide partial results useful for a more fine-grained analysis. We validate this evaluation protocol by showing it correlates well with human evaluation.

LeanEuclid: Formalizing Proofs and Diagrams. We construct *LeanEuclid*, a benchmark for testing machine learning on autoformalizing Euclidean geometry. As in Fig 1 (*Left*), each example in LeanEuclid has an informal theorem, proof, and diagram in LaTeX, as well as a formal theorem and proof in Lean. Data examples in LeanEuclid are manually formalized into Lean from Euclid's *Elements* (Heiberg,

Euclid uses the intersection of two circles (C) without proving its existence. Most readers would not find the proof problematic, as the two circles intersect in the diagram. Such implicit diagrammatic reasoning is ubiquitous in informal geometric proofs but needs to be handled explicitly in formal proofs (Beeson et al., 2019). Therefore, a naive attempt to autoformalize the proofs would be difficult, as it requires the model to fill in many diagrammatic reasoning gaps, with nothing to reference in the informal texts.

To mitigate diagrammatic gaps, LeanEuclid adopts a formal system named E (Avigad et al., 2009), introduced by philosophers for modeling diagrammatic reasoning in Euclid's *Elements*. It teases out a set of diagrammatic rules so that diagrammatic reasoning can be modeled as logical deductions. We implement E in Lean and provide proof automation to fill in diagrammatic reasoning gaps, using the same symbolic reasoning engine developed for equivalence checking. Our system enables formalizing all 48 theorems

RESEARCH

AlphaGeometry: An Olympiad-level AI system for geometry

17 JANUARY 2024

Trieu Trinh and Thang Luong

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More conclusions

- When you look at any piece of mathematics and think about how it works, you notice interesting things.
- Understanding how mathematics works is useful for:
  - mathematics
  - philosophy of mathematics
  - history of mathematics
  - cognitive science
  - education
  - automated reasoning
  - AI.
- It's also deeply satisfying.