Automated Exchange Economies

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Making Markets via Smart Contracts

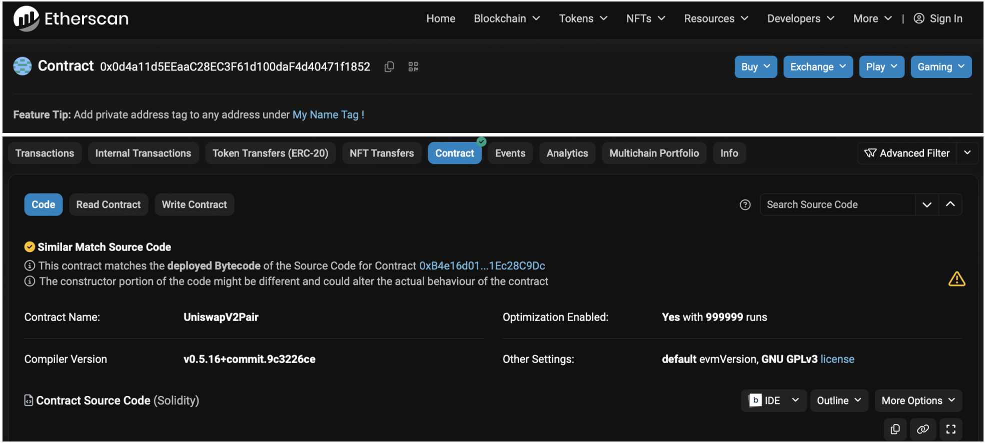
- How to design a "centralized" exchange on a distributed ledger?
  - Key friction: verifiable communications are (typically) costly
  - Suggests limit order books may be impractical
- Existing solution: ad hoc pricing functions called automated market makers
- Our research: establish a framework to evaluate how AMMs support liquidity provision and exchange

Making Markets via Smart Contracts

- An Automated Market Maker is a Smart Contract
  - Smart contract $\Leftarrow$ deterministic, verifiable script on a blockchain

- AMM Smart Contract has two key functions:
  1. Liquidity Provision Rules
     - LPs deposit or withdraw a portfolio of tokens:
     - Deposit (Mint): $(+e_a, +e_b)$ or Withdraw (Burn): $(-e_a, -e_b)$
  2. Liquidity Taking Rules:
     - LTs swap tokens at some pre-specified schedule
     - e.g. Swap a for b: $(+q_a, -q_b)$

Making Markets via Smart Contracts



Making Markets via Smart Contracts

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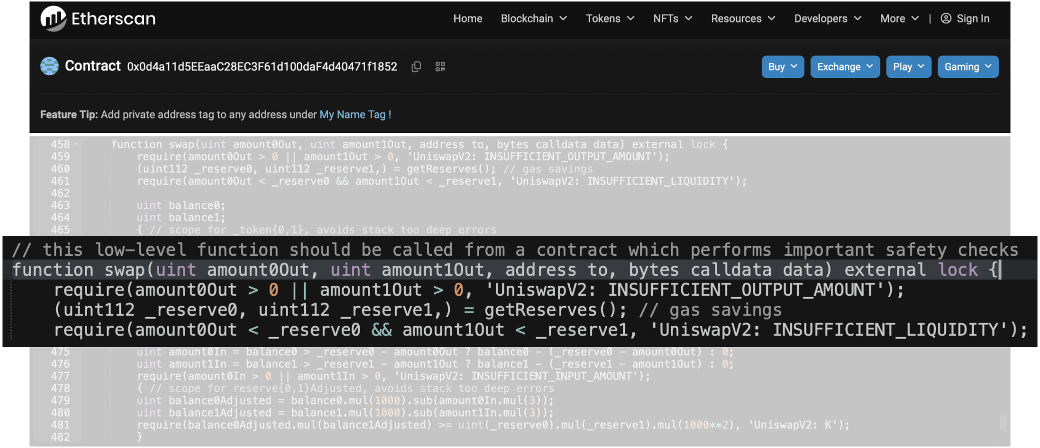
Contract 0x0d4a11d5EEaaC28EC3F61d100daF4d40471f1852 — Buy | Exchange | Play | Gaming

Feature Tip: Add private address tag to any address under My Name Tag !

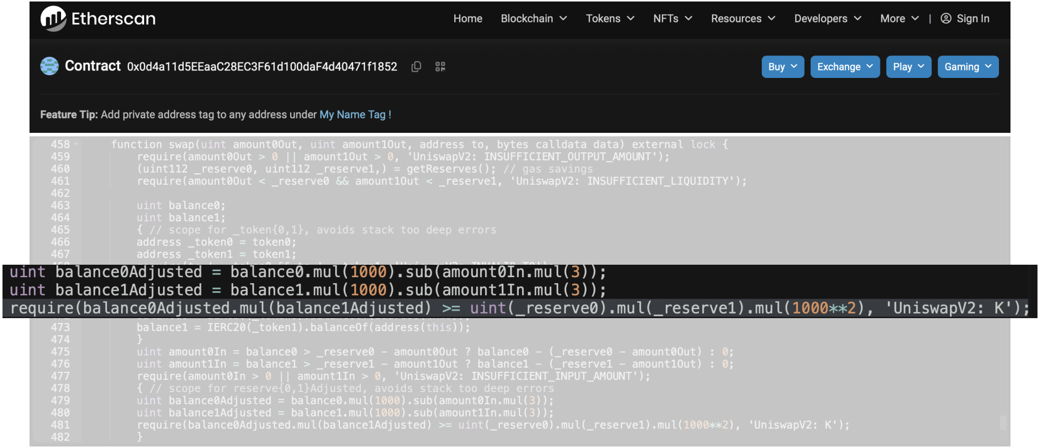
```
    function swap(uint amount0Out, uint amount1Out, address to, bytes calldata data) external lock {
        require(amount0Out > 0 || amount1Out > 0, 'UniswapV2: INSUFFICIENT_OUTPUT_AMOUNT');
        (uint112 _reserve0, uint112 _reserve1,) = getReserves(); // gas savings
        require(amount0Out < _reserve0 && amount1Out < _reserve1, 'UniswapV2: INSUFFICIENT_LIQUIDITY');

        uint balance0;
        uint balance1;
        { // scope for _token{0,1}, avoids stack too deep errors
        address _token0 = token0;
        address _token1 = token1;
        require(to != _token0 && to != _token1, 'UniswapV2: INVALID_TO');
        if (amount0Out > 0) _safeTransfer(_token0, to, amount0Out); // optimistically transfer tokens
        if (amount1Out > 0) _safeTransfer(_token1, to, amount1Out); // optimistically transfer tokens
        if (data.length > 0) IUniswapV2Callee(to).uniswapV2Call(msg.sender, amount0Out, amount1Out, data);
        balance0 = IERC20(_token0).balanceOf(address(this));
        balance1 = IERC20(_token1).balanceOf(address(this));
        }
        uint amount0In = balance0 > _reserve0 - amount0Out ? balance0 - (_reserve0 - amount0Out) : 0;
        uint amount1In = balance1 > _reserve1 - amount1Out ? balance1 - (_reserve1 - amount1Out) : 0;
        require(amount0In > 0 || amount1In > 0, 'UniswapV2: INSUFFICIENT_INPUT_AMOUNT');
        { // scope for reserve{0,1}Adjusted, avoids stack too deep errors
        uint balance0Adjusted = balance0.mul(1000).sub(amount0In.mul(3));
        uint balance1Adjusted = balance1.mul(1000).sub(amount1In.mul(3));
        require(balance0Adjusted.mul(balance1Adjusted) >= uint(_reserve0).mul(_reserve1).mul(1000**2), 'UniswapV2: K');
        }
```

Making Markets via Smart Contracts



Making Markets via Smart Contracts



Making Markets via Smart Contracts

- Liquidity Taking Rules:
  - Swap a for b: $(+q_a, -q_b)$
  - Rule implemented as function embedded in smart contract
  - Price schedule defined by "Constant Product Rule":

$$(e_a + q_a)(e_b - q_b) = e_a e_b$$

  - Slope of schedule defines implicit relative price of token b for a

Making Markets via Smart Contracts

- Questions
  - How should LPs choose deposits on AMMs?
  - How does design of the price schedule impact gains to trade between LPs and LTs?

- This paper:
  - Develop simple, tractable economic framework to answer these questions
  - Findings:
    - Adverse selection distorts intermediation *quantities* rather than *prices*
    - Typically suboptimal for LPs to deposit tokens in equal values as conventionally suggested
    - Efficiency of price function: trade-off between volume and adverse selection

Related Literature

- AMM Price "discovery"
  - **Do AMM prices reflect "true" prices?**
  - Angeris and Chitra (2020), Angeris et al (2021), Aoyagi (2022)

- AMM Liquidity
  - **What are the costs of creating AMM liquidity?**
  - Capponi and Jia (2021), Milionis et al (2022), Hasbrouck, Rivera, and Saleh (2022), Lehar and Parlour (2023), Fabi and Prat (2023)

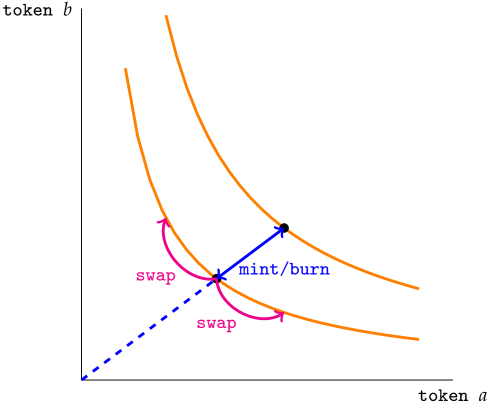
- AMM Design
  - **What is the optimal price function?**
  - Park (2022), Bergault et al (2023), Goyal et al (2023), Milionis, Moallemi, and Roughgarden (2023)

Active Liquidity Management

Liquidity Providers

- Industry/Literature defines liquidity providers as *passive*
  1. Interact with contract infrequently
  2. Only use Deposit/Withdraw functions

- What does the data say?



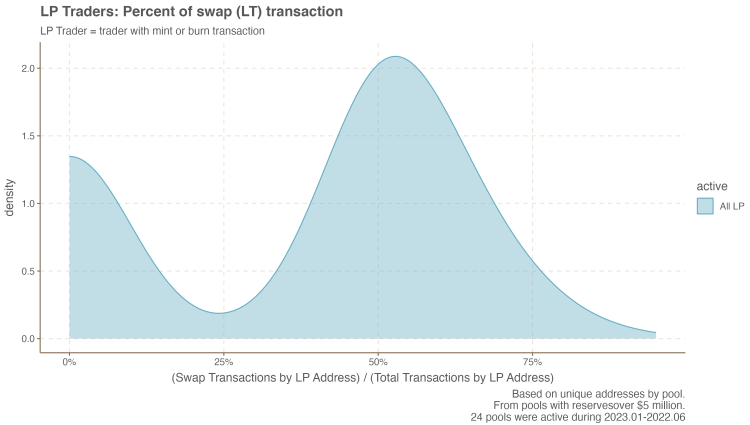
Liquidity Providers are Infrequent but "Active"

Uniswap Transaction Counts 2023-01-01 – 2023-06-30				
trader	burns	mints	swaps	total
LP	5,375	24,838	5,693	35,906
LT	0	0	1,252,596	1,252,596
Total	5,375	24,838	1,258,289	1,288,502

From pools with reserves over $5 million.
24 pools were active during this time period

- LPs have few interactions with contract relative to non-LPs
- LPs do use both functionalities
  - ☞ LP actions impact exchange prices

Liquidity Providers are Infrequent but “Active”



- By address: some LPs use Swap Functions, some do not

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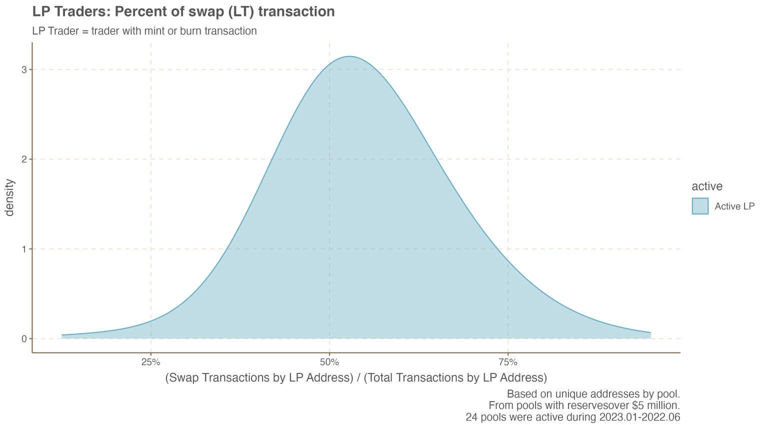

- By address: some LPs use Swap Functions, some do not
- Among active LPs, swaps make up large portion of activity

Liquidity Providers are Heterogeneous

Uniswap **TRADER** Counts 2023-01-01 – 2023-06-30				
	Unique Traders	Total Transactions	Liquidity Provisions	Liquidity **Takings**
LP active	1,854	10,069	43.5%	**56.5%**
LP passive	940	25,813	100.0%	**0.0%**

Based on unique addresses by pool From pools with reserves over $5 million. 24 pools were active during this time period

- Some LP addresses are passive and some are active
- Our paper addresses behavior of active LPs

Environment

Environment

- 2-by-2 economy (2 agents, 2 assets) in finite time
- Two risk-neutral agents:
  - Alice (LP) owns endowments (E_a, E_b) of a pair of tokens a and b
  - Bob (LT) may trade using the AMM (large number of "Bob"s)
- Timing in each period
  1. LP deposits tokens with exchange
  2. Public information about assets realized
  3. LT trades at exchange

Assets and Information

- Tokens $i \in \{a, b\}$ yield terminal value $\exp(d_{i,T})$ where

$$d_{i,T} = \sum_{t=0}^{T} y_{i,t} + \epsilon_i$$

  - Interpret $\exp(d_{i,T})$ as future "price" or service flow from the token
  - Residual independent uncertainty realized at T: $\mathbb{E}[\exp(\epsilon_i)] = 1$
  - Public information $y_{i,t}$ arrives each period:
    - $y_{i,t} = 0$ with prob $\hat{\pi}$, $y_{i,t} = -\Delta_l$ or $+\Delta_h$ with prob $(1 - \hat{\pi})/2$
    - Beginning of period beliefs

$$\mu_{i,t} = \mathbb{E}[\exp(d_{i,T})|y_0, \ldots, y_{t-1}] = \mathbb{E}_t[\exp(d_{i,T})]$$

Information, Assets, and Preferences

- LP makes deposits with expected valuation $\mu_{i,t} = \mathbb{E}_t[\exp(d_{i,T})]$

- LT trades with expected valuation $\hat{\mu}_{i,t} = \mathbb{E}_{t+1}[\exp(d_{i,T})] \exp(\eta_i)$
  (η_i is a preference shock)

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- LT trades with expected valuation $\hat{\mu}_{i,t} = \mathbb{E}_{t+1}[\exp(d_{i,T})]\exp(\eta_i)$ (η_i is a preference shock)

  ☞ Expositional assumption

    - If $y_{i,t} \in \{-\Delta_l, \Delta_h\}$ (for some i) then $\eta_a = \eta_b = 0$

  - Information event ($y_{i,t} \in \{-\Delta_l, \Delta_h\}$ some i) $\Rightarrow$ **pure informed trading** event
  - No information ($y_{a,t} = y_{b,t} = 0$) $\Rightarrow$ **pure taste/noise trading** event
  - LP trades-off losses from informed trading with gains from noise trading

LT's Problem

- Bob/LT faces a price schedule and maximizes expected dividends:

$$\max_{q_a,q_b} -\hat{\mu}_{a,t}q_a + \hat{\mu}_{b,t}q_b$$

subject to

$$(e_{a,t} + q_a)(e_{b,t} - q_b) = e_{a,t}e_{b,t}$$

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- Optimality implies

$$\frac{\hat{\mu}_{b,t}}{\hat{\mu}_{a,t}} = \frac{e_{a,t} + q_a}{e_{b,t} - q_b} \equiv \frac{x_{a,t}}{x_{b,t}}$$

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- Impose this behavior and examine Alice/LP's optimal choice of deposits

  ☞ Alice/LP's ex-post allocation satisfies:

$$x_{a,t} x_{b,t} = e_{a,t} e_{b,t}, \qquad \hat{\mu}_{a,t} x_{a,t} = \hat{\mu}_{b,t} x_{b,t}$$

LP's Dynamic Problem

- Assume probability of pure noise trade event is π and pure informed trade is $1-\pi$

$$V_T(E_a, E_b, \vec{\mu}_T) = \mu_{a,T}E_a + \mu_{b,T}E_b$$

$$V_t(E_a, E_b, \vec{\mu}_t) = \max_{e_a, e_b} \pi \mathbb{E} V_{t+1}(E'_a, E'_b, \vec{\mu}_{t+1}) + (1-\pi)\mathbb{E} V_{t+1}(E'_a, E'_b, \vec{\mu}_{t+1})$$

$$\begin{array}{lll} \textit{with} & E'_a = E_a - e_a + x_a & \textit{Accounting} \\ & E'_b = E_b - e_b + x_b & \\ & & \\ & \mu_{t+1} = \mu_t \quad \textit{if } y_t = 0 & \textit{Beliefs} \\ & \mu_{t+1} = \hat{\mu}_t \quad \textit{if } y_t \neq 0 & \\ & & \\ & e_a e_b = x_a x_b & \textit{Constant Product} \\ & \hat{\mu}_{a,t} x_a = \hat{\mu}_{b,t} x_b & \textit{Bob's optimality} \end{array}$$

- Rest of talk focus on one-shot game (drop t subscripts)

Optimal Liquidity Provision

LP's Problem

- LP's one-shot problem

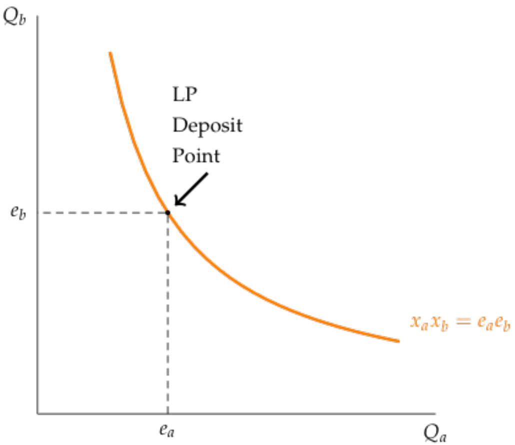
$$\max_{e_a,e_b} \pi \sum_i \mu_i \mathbb{E}[x_i - e_i] + (1-\pi) \sum_i \mathbb{E}[\hat{\mu}_i (x_i - e_i)]$$

subject to

$$x_a x_b = e_a e_b, \qquad \hat{\mu}_a x_a = \hat{\mu}_b x_b, \qquad 0 \leq e_j \leq E_j$$

  - LPs deposit choice influences shape and position of pricing curve

AMM Economics in a Graph



- CPMM implicitly defines relative price of tokens for LTs

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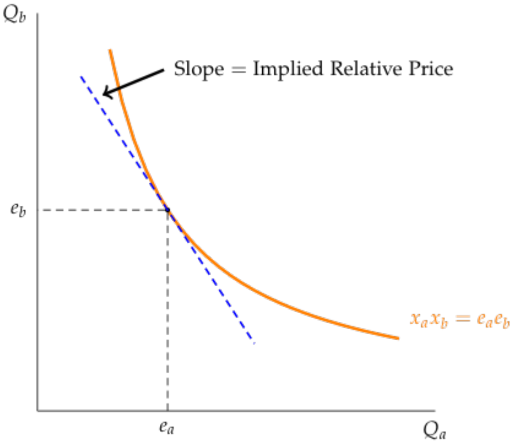

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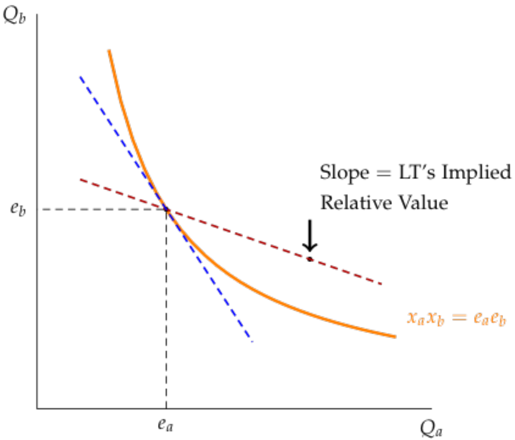

- Bob (LT) trades if relative valuation is different from CPMM implicit relative price

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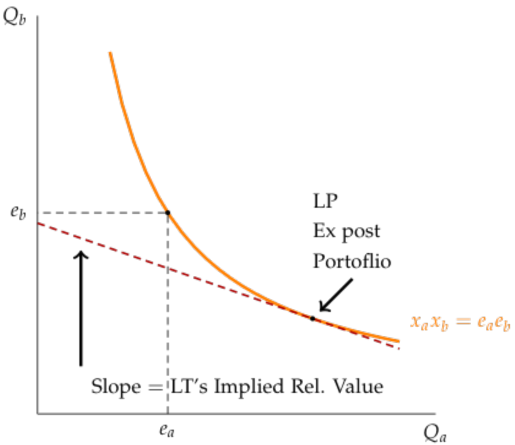

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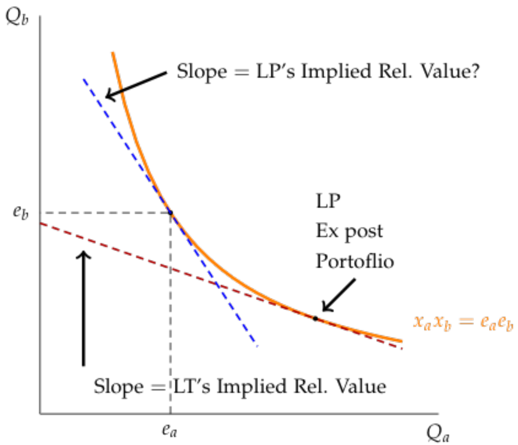
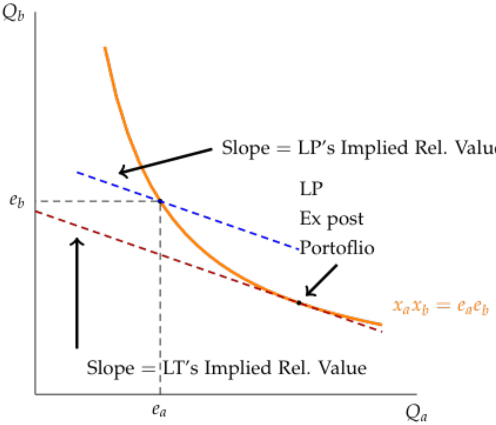


- Alice (LP) gains if relative valuation close to initial CPMM implicit relative price

AMM Economics in a Graph



- Alice (LP) loses if (ex post) relative valuation is similar to that of Bob (LT)

LP's Problem

- Re-write LP's problem

$$\max_{e_a, e_b} \left[\pi\gamma_U + (1-\pi)\gamma_I\right] \sqrt{\mu_a e_a}\sqrt{\mu_b e_b} - \left(\sqrt{\mu_a e_a} - \sqrt{\mu_b e_b}\right)^2$$

where

  - γ_U, γ_I functions of distributions of belief dispersion $H(\mu_i/\hat{\mu}_i)$
  - $\gamma_U > 0$ and $\gamma_I < 0$

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  - When gains to LP, deviation from equal-value deposit yields first order gains and second order losses

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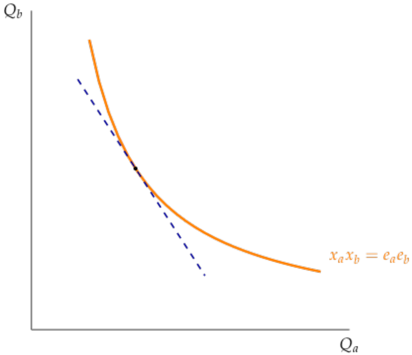
  - $\gamma_U > 0$ and $\gamma_I < 0$
  - Gains to LP only when π is large enough
  - When gains to LP, deviation from equal-value deposit yields first order gains and second order losses
  - Revision to conventional wisdom:
    - ☞ "LPs should deposit in equal values only if no gains to trade in market"

LP's Problem: A Simple Rule

- With only uninformed trading, easy for LP to guarantee no losses



- Tangency and constant product implies $\mu_a e_a = \mu_b e_b$

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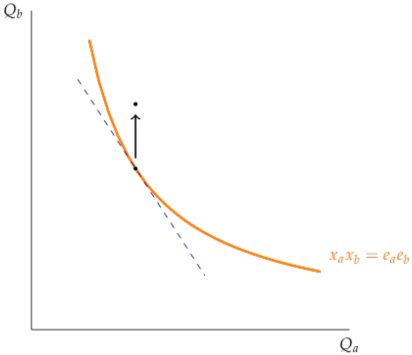

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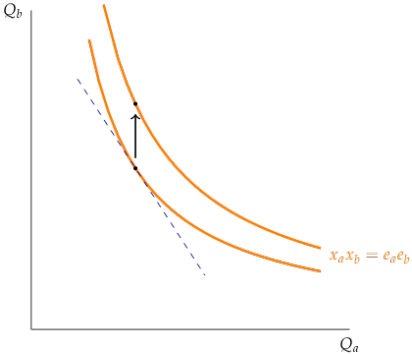

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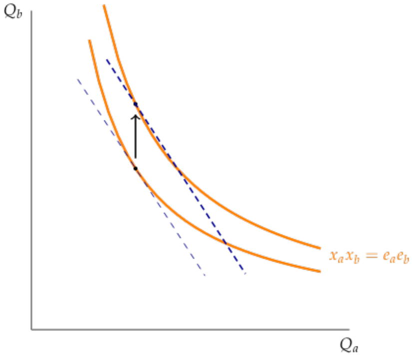


- Tangency and constant product implies $\mu_a e_a = \mu_b e_b$
- Small deviations yield second order losses around the deposit point but first order gains for larger trades

Optimal Liquidity Provision

Proposition (*Optimal Liquidity*)

The optimal liquidity deposit with π proportion of uninformed trading and $1-\pi$ proportion of informed trading satisfies

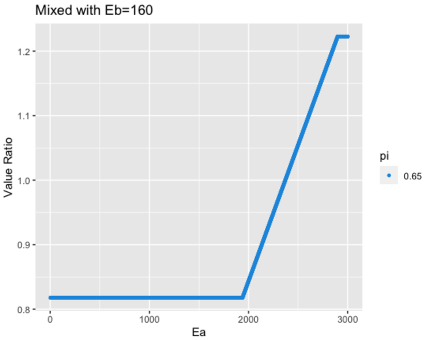
$$e_a^* = E_a, e_b^* = \min\left\{\left(\frac{\pi}{2}\left(\mathbb{E}_U[\omega] + \mathbb{E}_U\left[\frac{1}{\omega}\right]\right) + (1-\pi)\mathbb{E}_I[\psi]\right)^2 \frac{\mu_a}{\mu_b} E_a, E_b\right\}, \text{ if } \mu_a E_a \leq \mu_b E_b$$

and

$$e_a^* = \min\left\{\left(\frac{\pi}{2}\left(\mathbb{E}_U[\omega] + \mathbb{E}_U\left[\frac{1}{\omega}\right]\right) + (1-\pi)\mathbb{E}_I[\psi]\right)^2 \frac{\mu_b}{\mu_a} E_b, E_a\right\}, e_b^* = E_b, \text{ if } \mu_a E_a > \mu_b E_b$$

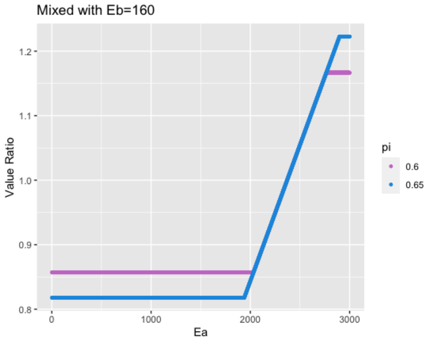
- Linear preferences $\Rightarrow$ expect (and find) corner solutions

Optimal Liquidity: Comparative Statics



1. *Change in Endowments:* Value ratio $\mu_a e_a / \mu_b e_b$ rises with E_a

Optimal Liquidity: Comparative Statics



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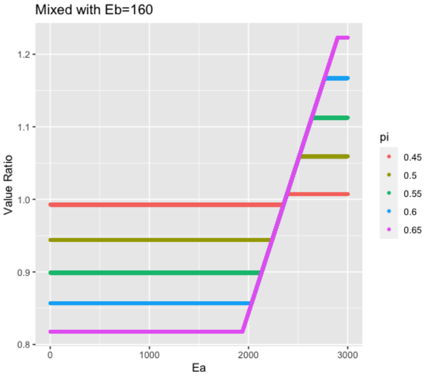
Optimal Liquidity: Comparative Statics



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☞ Adverse Selection distorts intermediation quantities

Efficiency

Implications for AMM Design

- How should the price schedule be designed?
- Framework offers a new tradeoff:
  - Convexity hinders trading volume and reduces realized gains to trade
  - Convexity offers protection from informed trading

Local Convexity of the Price Function

- Consider a class of of price functions that differ by local convexity:

$$(e_a + (1-\tau)q_a)(e_b - (1-\tau)q_b) = e_a e_b$$

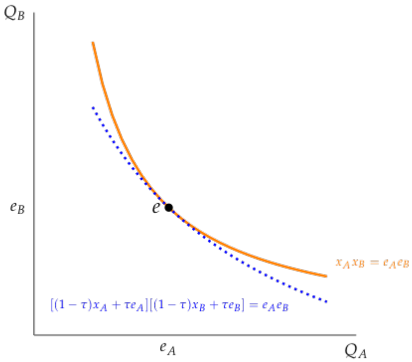
- Re-write in ex post portfolios for LP

$$((1-\tau)x_a + \tau e_a)((1-\tau)x_b - \tau e_b) = e_a e_b$$

Lemma

If LT's beliefs are bounded, there exists $\delta > 0$ such that for all $\tau \leq \delta$, the LP's optimal deposit does not vary with τ.

Local Convexity of the Price Function



- Increasing τ lowers convexity locally (more linear) around LP's deposit choice

Local Convexity of the Price Function

Proposition (*Efficient Price Design*)

If the LT's beliefs $\hat{\mu}_i$ are bounded, then for any convex, smoothly decreasing price function $G(\cdot)$, there exists $\delta > 0$ such that for $\tau < \delta$ the price function implicitly defined by

$$(1-\tau)\,y + \tau e_b = G((1-\tau)\,x + \tau e_a)$$

increases both the LP's and the LT's expected returns proportionally by $\frac{\tau}{1-\tau}$.

- Convex prices limit trade
- If liquidity provision profitable with convex prices, increasing trade is profitable

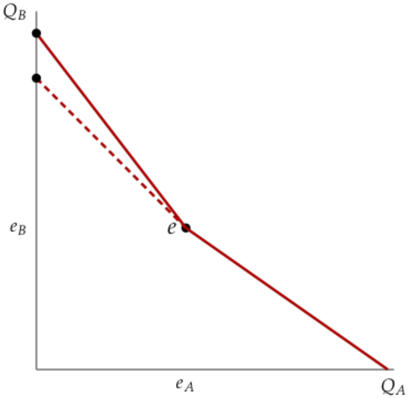
☞ A small reduction in (local) convexity raises market efficiency

Global Convexity of the Price Function

- *Claim:* If "extreme" beliefs possible, reducing local convexity is not always optimal
- Intuition:
  - Reducing local convexity promotes more (extensive) trading volume (volume effect)
  - Reducing local convexity implies lower prices for extreme trades (price effect)
  - At globally linear prices, price effect dominates, reduces profits
    - Easy to show using piece-wise linear approximation to the price function
  - Adverse selection strengthens this results

☞ Globally linear prices are not efficient

Global Convexity of the Price Function



- Consider how reducing local convexity (around (e_a, e_b)) impacts profits at the boundary

Global Convexity of the Price Function

- Let $-p_h$ be the slope of the price function for $x_a < e_a$
- Linear pricing $\Rightarrow$ LTs trade to the boundary if $\hat{\mu}_a / \mu_a > p_h$
- Marginal effect on profits from reducing $|p_h|$:
  - Reduces prices for all uninformed LTs who trade: $-[1 - F(p_h)]$
  - Increases volume with uninformed LTs: $+(p_h - 1)f(p_h)$
  - Reduces prices for all informed LTs who trade: $-[1 - F(p_h)]$
- Net effect strictly negative as $p_h \to 1$

$$-[(1 - F(p_h)) - \pi((p_h - 1)f(p_h)]$$

☞ Globally linear prices are not efficient

Wrap Up

To do

- Exploring consequences of CPMM for allocation and gains to trade
- Extending analysis to dynamic framework
- Connecting model gains to trade to empirics from existing AMMs
- Use framework to conduct Robust Mechanism Design for AMMs

Appendix