The Cantor Schroeder-Bernstein Theorem

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Abstract

The Cantor Schroeder-Bernstein Theorem states that if there exists an injection $f\colon A \to B$ and an injection $g\colon B \to A$, then there exists a bijection $h\colon A \to B$. This is an intuitive result, but its proof is surprisingly tricky.

In this article, we briefly review injections, surjections, and bijections, and use these basic facts to support a proof of the Cantor Schroeder-Bernstein Theorem.

Much of this article was derived from https://web.williams.edu/Mathematics/lg5/CanBer.pdf. Here we just fill in all of the details.

1 Preliminaries

This should mostly be review. You should work out proofs of each of these theorems yourself.

1.1 Functions

Definition 1.1 (Injection).

A function $f\colon A \to B$ is **injective** if and only if $f(x) = f(y)$ implies $x = y$.

In other words, if the outputs are the same, the inputs must be the same.

Definition 1.2 (Surjection).

A function $f\colon A \to B$ is **surjective** if for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Definition 1.3 (Bijection).

A function $f\colon A \to B$ is a bijection if it is an injection and it is a surjection.

Prove the following theorem.

Theorem 1.4 (Bijectivity is an equivalence relation).

We say that $A \sim B$ if and only if there exists a bijection $f\colon A \to B$. Then $\sim$ is an **equivalence relation**. That is,

1. Reflexivity: $A \sim A$.
2. Symmetry: $A \sim B$ implies $B \sim A$.
3. Transitivity: $A \sim B$ and $B \sim C$ implies $A \sim C$.

1.2 Unions and Intersections

Definition 1.5 (Union).

Consider a (possibly infinite) series of sets A_i. Then $\bigcup A_i$ is the set of elements contained in at least one A_i.

Definition 1.6 (Intersection).

Consider a (possibly infinite) series of sets A_i. Then $\bigcap A_i$ is the set of elements contained in every A_i.

From here on out we will commit an abuse of notation. Given a function $f\colon A \to B$ and $A_0 \subseteq A$, we write $f(A_0)$ to denote the **range** of f on A_0. In other words, $f(A_0)$ (which is a subset of B) is the set of elements b such that there exists some element $a \in A_0$ where $f(a) = b$.

The following theorem will be useful for the last part of our proof.

Theorem 1.7.

Given a set A, a series of sets $A_i \subseteq A$, and an injective function $f\colon \bigcup A_i \to B$,

$$f(\bigcup A_i) = \bigcup f(A_i)$$
$$f(\bigcap A_i) = \bigcap f(A_i)$$

2 Cantor Schroeder-Bernstein

2.1 Groundwork

Define sequences A_n and B_n recursively as follows:

1. $A_0 = A$
2. $B_0 = B$
3. $A_n = g(B_{n-1})$ for $n \geq 1$
4. $B_n = f(A_{n-1})$ for $n \geq 1$

The following two lemmas are the meat of the proof.

Lemma 2.1. For $n \geq 0$, $A_n \sim B_{n+1}$.

Proof. Recall that $B_{n+1} = g(A_n)$. Note that $g\colon A_n \to B_{n+1}$ is injective as g is injective on the entirety of A. And since $B_{n+1} = \operatorname{range} A_n$ by definition, g is also surjective. Thus $g\colon A_n \to B_{n+1}$ is a bijection. $\square$

By symmetry, we have $B_n \sim A_{n+1}$ as well.

Lemma 2.2. For $n \geq 0$, $A_n \supseteq A_{n+1}$ and $B_{n+1} \supseteq B_n$.

Let us first look at a few small cases to gain some intuition. Note that

$$(A_0, A_1, A_2, A_3) = (A, gB, gfA, gfgB).$$

Notice for every term except for A_0, there is a g on the "outside". So if we can show inclusion on the sets g is being applied on, then we can also show inclusion on the result after g is implied.

As a concrete example,

$$B \supseteq fA \implies gB \supseteq gfA.$$

Proof. We induct on n. The base case is straightforward, so we omit it.

Recall that $A_n = g(B_{n-1})$ and $A_{n+1} = g(B_n)$. Since $B_{n-1} \supseteq B_n$, we conclude that $g(B_{n-1}) \supseteq g(B_n)$. By symmetry, we have $B_n \supseteq B_{n+1}$ as well. $\square$

2.2 Using our lemmas

Now there are two cases. Either there exists some n such that $A_n = A_{n+1}$ or $B_n = B_{n+1}$, in which case we are done, or there does not. The full details of the first case are left to Appendix A, but here is a general sketch. We have $A_n \sim A_{n+1} \sim B_n$, where the important part is $A_n \sim B_n$. We can show that $A_n \sim B_n \implies A_{n-1} \sim B_{n-1}$, which eventually cascades to $A_0 \sim B_0$.

Now suppose that there exists no n such that $A_n = A_{n+1}$ or $B_n = B_{n+1}$. Then we can rewrite Lemmas 2.2 and 2.1 as follows.

Lemma 2.3. For $n \geq 0$, $A_n \supsetneq A_{n+1}$ and $B_n \supsetneq B_{n+1}$.

Then define $A_n^\star$ as $A_n - A_{n+1}$ (where $-$ is set subtraction). Define $B^\star$ similarly. Note that $A_n^\star$ is never empty as the inclusions in Lemma 2.3 are strict and A_n is never empty. This is a simple proof by induction; full details in Appendix B.

Lemma 2.4. For $n \geq 0$, $A_n^\star \sim B_{n+1}^\star$.

The proof is left to Appendix C.

Lemma 2.5. There exists a bijection $h_0 \colon \bigcup A_i \to \bigcup B_i$.

This is a consequence of Lemma D.1.

Let's take stock of where we are. We have bijected most of A to most of B, and with Lemma D.1 we have a tool to compose bijections of disjoint unions. So all we have to do is answer the following questions:

1. What part of A is not in $\bigcup A_i^\star$?
2. How do we biject it to its counterpart in B?

Lemma 2.6. The disjoint union of $\bigcup A_i^\star$ and $\bigcap A_i$ is A.

Proof. Note that $a \in A$ is in $\bigcup A_i^\star$ if and only if there exists some n such that $a \in A_n$ but $a \notin A_{n+1}$. If there exists no such n, then because $a \in A_0$, we conclude a is in every A_i. In other words, $a \in \bigcap A_i$. $\square$

Lemma 2.7. The function f is a bijection from $\bigcap A_i$ to $\bigcap B_i$.

Proof. Note by Theorem 1.7 that

$$f(\bigcap A_i) = \bigcap B_{i+1},$$

and $B_0 \bigcap B_{i+1} = \bigcap B_i$ as every B_i is a subset of B_0.

So the range of $f(\bigcap A_i)$ is exactly $\bigcap B_i$, meaning that $f \colon \bigcap A_i \to \bigcap B_i$ is a surjection. Since f is injective by definition, f is a bijection, as desired. $\square$

To finish, note that by Lemma D.1, $\bigcup A_i^\star \sim \bigcup B_i^\star$ and $\bigcap A_i \sim \bigcap B_i$ implies $A \sim B$.

A Proof of non-strict inclusion case

Here we handle the full details of the non-strict inclusion case as a separate theorem.

Theorem A.1.

For any $n \geq 0$, $A_n = A_{n+1} \implies A_0 \sim B_0$.

If we show this, we show by symmetry that $B_n = B_{n+1} \implies A_0 \sim B_0$.

Lemma A.2. For any $n \geq 0$, $A_n \sim B_n \implies A_0 \sim B_0$.

Proof. We proceed by induction on n. The base case of $n = 0$ is obvious.

Now suppose $A_n \sim B_n$; we want to show that $A_{n+1} \sim B_{n+1}$. But by Lemma 2.1,

$$B_n \sim A_{n+1} \sim B_{n+1} \sim A_n,$$

which implies that $A_0 \sim B_0$. □

Note that $A_n \sim A_{n+1} \sim B_n$ by Lemma 2.1, which implies $A_0 \sim B_0$ by Lemma A.2, as desired.

B A_n is non-empty

Lemma B.1. For all $n \geq 0$, A_n and B_n are non-empty.

Proof. We induct on n. This is obviously true for $n = 0$.[1]

Now suppose A_n and B_n are non-empty; we want to show that A_{n+1} and B_{n+1} are non-empty. But A_{n+1} is the range of $g(B_n)$, and since B_n is non-empty, all we must do to show A_{n+1} is non-empty is select an element in B_n.

Symmetrically, B_{n+1} is non-empty. □

C Proof of bijection between $A_n^\star$ and $B_{n+1}^\star$

Lemma C.1. For all $n \geq 0$, there is a bijection between $A_n - A_{n+1}$ and $B_{n+1} - B_{n+2}$.

Proof. Note that f bijects A_n to B_{n+1}, and furthermore, it also bijects A_{n+1} into B_{n+2}. Since $A_n \supseteq A_{n+1}$ and $B_n \supseteq B_{n+1}$,

$$f(A_n - A_{n+1}) = f(A_n) - f(A_{n+1}) = B_{n+1} - B_{n+2}.$$

Since f is injective, f bijects $A_n - A_{n+1}$ to $B_{n+1} - B_{n+2}$. □

D Disjoint union bijections

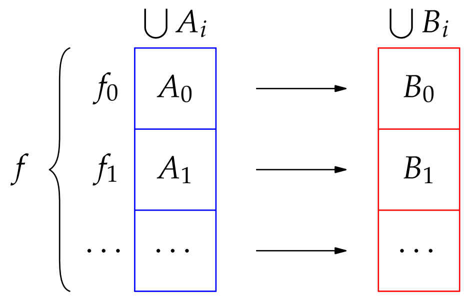
This is a lemma that is generally useful even outside of this specific proof.

Lemma D.1. Suppose there exists a sequence of pairwise disjoint sets A_i and another such sequence B_i. Then,

$$\bigcup A_i \sim \bigcup B_i.$$

Proof. Select a bijection $f_i \colon A_i \to B_i$ for each i. Then, we can explicitly construct a bijection $f \colon \bigcup A_i \to B_i$ as follows: if $a \in A_i$, then $f(a) = f_i(a)$. This is well defined because $a \in A_i$ for exactly one i.

[1] This is not strictly true, but the case where $A_0 = B_0 = \emptyset$ is so trivial that we don't care.



$\square$