A Proof of PIE

Dennis Chen

Fall 2023

Abstract

The Principle of Inclusion-Exclusion is perhaps one of the most famous theorems in combinatorics. Yet many students only know that $|S_1 \cup S_2| = |S_1| + |S_2| + |S_1 \cap S_2|$, and not what PIE actually is or how to prove it. We remedy this here: this document explains what the theorem states, what it really means, and why it is true.

Preliminary knowledge assumed:

1. A basic understanding of set theory.
2. Awareness of special cases of PIE and their proofs, e.g. $|A \cup B| = |A| + |B| - |A \cap B|$.

1 The Theorem

Theorem 1.1 (The Principle of Inclusion-Exclusion).

For any set $\mathcal{A}$ whose elements are all sets themselves,

$$\left|\bigcup_{S \in \mathcal{A}} S\right| = \sum_{\mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'|+1} \left|\bigcap_{S \in \mathcal{A}'} S\right|.$$

This is the most precise and compact way that PIE can be stated. Let's unravel it now!

1.1 The Left Hand Side

Since $\mathcal{A}$ has sets as its elements, we can write it in the form $\mathcal{A} = \{S_1, \dots, S_n\}$, where each S_i is a set. Then the left hand side describes something familiar: the number of (distinct) elements in at least one of $S_1, \dots, S_n$.

1.2 The Right Hand Side

First of all, what is the set of all possible $\mathcal{A}'$? It turns out that $\mathcal{A}'$ spans the set of all possible subsets of $\{S_1, \dots, S_n\}$. If you know what a **powerset** is, then the set of all possible $\mathcal{A}'$ is just the powerset of $\mathcal{A}$.

Let's ignore the factor of -1 for a second and focus on the $\bigcap$ mess. It's just the intersection of some of the sets in $\{S_1, \dots, S_n\}$. The way

$$\sum_{\mathcal{A}' \subseteq \mathcal{A}}$$

works, every possible intersection involving some combination of $\{S_1, \dots, S_n\}$ will be considered.

The factor of -1 essentially states:

1. We add the size of the intersection of an odd number of sets S_i,
2. and we subtract the size of the intersection of an even number of sets S.

To simplify, we will assume from now on that $\mathcal{A}'$ is non-empty when referring to

$$\left| \bigcap_{S \in \mathcal{A}'} S \right|.$$

It doesn't end up changing the sum, because the size of the union of no sets is (by definition) 0.

1.3 Examples of PIE

We analyze the 2 and 3 set cases of PIE to get an idea of what is going on.

Example 1.2 (PIE for Two Sets).

Suppose $\mathcal{A} = \{S_1, S_2\}$. Then PIE states

$$\begin{aligned} |S_1 \cup S_2| &= (-1)^{|\{S_1\}|+1}|S_1| + (-1)^{|\{S_2\}|+1}|S_2| + (-1)^{|\{S_1,S_2\}|+1}|S_1 \cap S_2| \\ &= |S_1| + |S_2| - |S_1 \cap S_2|. \end{aligned}$$

For the three set case, we are not going to explicitly show why the signs are what they are, mostly because the equation would be too long to display on one line.

Example 1.3 (PIE for Three Sets).

Suppose $\mathcal{A} = \{S_1, S_2, S_3\}$. Then PIE states

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_2 \cap S_3| - |S_3 \cap S_1| + |S_1 \cap S_2 \cap S_3|.$$

Make sure you understand how these specific cases of PIE are derived from the general theorem!

2 The Proof

2.1 Preliminaries

We first establish a few facts about sets.

Definition 2.1 (Disjoint Sets).

We say sets A and B are **disjoint** if $A \cap B = \{\}$; that is, the intersection of A and B is the empty set.

We define disjoint sets just to reduce the ratio of symbols to words in our proof.

Theorem 2.2.

If A and B are disjoint, then

$$|A \cap S| + |B \cap S| = |(A \cup B) \cap S|.$$

Theorem 2.3.

As a corollary, if sets $A_1, \ldots, A_n$ are pairwise disjoint, then

$$\sum_{i=1}^{n} |A_i \cap S| = \left| \left(\bigcup_{i=1}^{n} A_i \right) \cap S \right|.$$

Convince yourself both of these theorems are true.

2.2 Breaking it down into elements

Let

$$x_1, \ldots, x_m = \bigcup_{S \in \mathcal{A}} S.$$

In other words, let $x_1, \ldots, x_m$ be all the elements in at least one of $S_1, \ldots, S_n$. Note that m is the size of the union of $S_1, \ldots, S_n$.

If we can show that for each $x \in \{x_1, \ldots, x_m\}$,

$$\sum_{\mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'+1|} \left| \bigcap_{S \in \mathcal{A}'} S \cap \{x\} \right| = 1,$$

then by Theorem 2.3,

$$\sum_{\mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'+1|} \left| \bigcap_{S \in \mathcal{A}'} S \cap \{x_1, \ldots, x_m\} \right| = m.$$

Furthermore,

$$\bigcap_{S \in \mathcal{A}'} \subseteq \{x_1, \ldots, x_m\}$$

by definition, so we can simplify the equation further into

$$\sum_{\mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'+1|} \left| \bigcap_{S \in \mathcal{A}'} S \right| = m,$$

which is exactly what we want to show.

2.3 Looking at an individual element

So what were we really doing in the previous section? Intuitively, each term of the form

$$\sum_{\mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'+1|} \left| \bigcap_{S \in \mathcal{A}'} S \cap \{x\} \right|$$

represents the number of times that x is "counted" in the right-hand side of PIE. And really, the reason we use e.g. the two set case of PIE is to ensure each element is counted exactly once when looking at $|S_1 \cup S_2|$. We are doing the same thing here, but on a more general level.

Let's look at an arbitrary element of x from now on. Define $\mathcal{B}$ to be the subset of $\mathcal{A}$ such that

$$x \in S \iff S \in \mathcal{B}.$$

In other words, $\mathcal{B}$ is the set of every S such that x is an element of S.

Furthermore, note that

$$\left|\bigcap_{S\in\mathcal{A}'} S \cap \{x\}\right| = 1$$

if and only if $\{\} \subsetneq \mathcal{A}' \subseteq \mathcal{B}$. (Otherwise, the intersection has size 0.) So

$$\sum_{\mathcal{A}'\subseteq\mathcal{A}} (-1)^{|\mathcal{A}'+1|} \left|\bigcap_{S\in\mathcal{A}'} S \cap \{x\}\right|$$

is equivalent to the difference between the number of non-empty subsets of $\mathcal{B}$ with odd size and the number of non-empty subsets of $\mathcal{B}$ with even size.

At this point, I recommend you do at least one of the following exercises for intuition on the argument that is about to follow.

Problem 2.4. How many sequences of n coin flips contain an even number of heads?

Problem 2.5 (AIME 1983/13). For $\{1, 2, 3, \ldots, n\}$ and each of its non-empty subsets a unique **alternating sum** is defined as follows. Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. For example, the alternating sum for $\{1, 2, 3, 6, 9\}$ is $9 - 6 + 3 - 2 + 1 = 5$ and for $\{5\}$ it is simply 5. Find the sum of all such alternating sums for $n = 7$.

By definition, there is some element S_1 of $\mathcal{B}$, i.e. x is an element of some member of $\mathcal{A}$ — otherwise x would not be in $\{x_1, \ldots, x_m\}$. Note that every non-empty subset $\mathcal{A}'$ of $\mathcal{B}$ not containing S_1 can be paired off with the subset $\mathcal{A}' \cup \{S_1\}$. This leaves us with one unpaired subset: $\{S_1\}$.

Thus the number of non-empty subsets of $\mathcal{B}$ with odd size is exactly 1 greater than the number of non-empty subsets of $\mathcal{B}$ with even size, as desired.

2.4 Taking stock of it all

So what did we do in the proof?

1. We established in Subsection 2.1 that we can reasonably "break down" intersections.
2. In Subsection 2.2, we followed through with that idea by breaking down $\{x_1, \ldots, x_m\}$. (More accurately, we started with it "broken down", and showed that we can "stitch it back together".) We also used some wishful thinking: if each broken down sum was 1, then we could add all these sums together, element-by-element, until we got what we wanted.
3. Finally, we showed in Subsection 2.3 that each broken down sum was indeed 1.