Name: ______________________________

Andrew ID: ________________________

General Instructions:

- This exam prep contains two sections: Section A includes computational problems, and Section B includes conceptual and proof-based problems. Please read the instructions at the beginning of each section carefully.
- You are allowed a full page of notes (front and back).
- No calculators or any other electronics are permitted.
- Write your answers clearly and make sure your handwriting is legible. If we cannot read your work, we will not grade the problem.
- All work must be completed in the space provided. If you need scratch paper, there is some at the front of the class. Please note that scratch work will not be graded.
- Please ask questions if anything is unclear.
- If you finish early, check all of your work, then bring this packet up to the front of the class.
- Good luck!

Section A.

Instructions:

1. Each problem in this section is worth **one point** and no partial credit will be given.
2. You do not need to show your work or provide justification on any problem in Section A.
3. Make sure to **clearly box or circle your answer**.

A1. Find all solutions to the vector equation

$$x\begin{pmatrix}1\\2\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix}+z\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}.$$

Solution.

$$x = 1+t$$
$$y = -2-3t$$
$$z = t$$

for any real number t.

A2. Let

$$\vec{v}_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\vec{v}_2=\begin{pmatrix}0\\1\\1\end{pmatrix}\ \vec{v}_3=\begin{pmatrix}2\\3\\a\end{pmatrix}.$$

Find all values of a so that $\mathrm{Span}(\vec{v}_1,\vec{v}_2,\vec{v}_3)=\mathbb{R}^3$.

Solution. All real numbers except for $a=3$.

A3. Let A be the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Write the solution set to the matrix-vector equation $A\vec{x} = \vec{0}$ as $\mathrm{Span}(\vec{u}, \vec{v})$ for some vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^4$.

Solution.

$$\mathrm{Span}\left(\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

Section B.

INSTRUCTIONS:

1. Each problem in this section is worth **5 points**.
2. You must provide justification for all of your answers in Section B.
3. Points will be awarded based on the rubric below. Note that half points may be awarded, and further rubric items may be added to cover potential cases not outlined below.

Points	**Rubric**
5	Solution is presented with clear justification that is logically complete and correct. May include minor typos and computational errors if they do not majorly impact the argument. No important steps are missing or assumed. All assumptions and special cases have been covered. All suggestions for improvement come under the category of "improvements for clarity" rather than "correcting logical errors". Omission of details will be judged depending on context of the material, with simpler steps being acceptable for omission when covering more advanced topics.
4	Solution is close to full and complete, but contains either a computational error or an error in reasoning that majorly impacts the argument. This score is also appropriate for solutions that are mathematically sound but confusingly written.
3	Solution is incorrect, but understanding of the problem was demonstrated and student provided a clear outline of a potential approach with information about where they got stuck **-or-** solution is correct but justification is insufficient or so confusingly written that it cannot be followed with a reasonable amount of effort.
2	Solution is incorrect, but student demonstrated understanding of the problem **-or-** solution is correct and student did not provide justification for their answer.
1	Solution is incorrect and student did not demonstrate understanding of the problem, but did demonstrate some knowledge of relevant material.
0	Solution is incorrect or incomplete, and there was no demonstration of knowledge of relevant material.

B1. List all possible reduced row echelon forms of a 3×4 matrix with a pivot in exactly two rows. Use "$*$" to denote entries which can be equal to any real number. Make sure to justify how you know you've checked all possible cases.

Solution. Note that a row without a pivot must contain all zeros. So, we are looking for matrices with the bottom row containing only zero entries, and two pivots in the first two rows. We have the following list of possibilities.

Case 1- Pivot in first column

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Case 2 - Pivot in second column

$$\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Case 3- Pivot in third column

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

B2. True or False: A HOMOGENEOUS system of 3 linear equations in 5 variables has a nonzero solution. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample**.

Solution. This is true. Let A denote the coefficient matrix of this system. Since A has 3 rows and 5 columns, A must have a column with no pivot. Since we know homogeneous systems are always consistent, this implies that the system has infinitely many solutions, and so one of these solutions must be nonzero.

B3. True of False: If the system $A\vec{x} = \vec{b}$ has exactly one solution for a given $\vec{b}$, then the only solution to $A\vec{x} = \vec{0}$ is the zero vector. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample**.

Solution. This is true. Since the system $A\vec{x} = \vec{b}$ has exactly one solution, we know that the matrix A must have a pivot in each column. Since we know that $A\vec{x} = \vec{0}$ is always consistent, it has exactly one solution.