Name: ______________________________

Andrew ID: ________________________

General Instructions:

- This exam prep contains two sections: Section A includes computational problems, and Section B includes conceptual and proof-based problems. Please read the instructions at the beginning of each section carefully.
- You are allowed a full page of notes (front and back).
- No calculators or any other electronics are permitted.
- Write your answers clearly and make sure your handwriting is legible. If we cannot read your work, we will not grade the problem.
- All work must be completed in the space provided. If you need scratch paper, there is some at the front of the class. Please note that scratch work will not be graded.
- Please ask questions if anything is unclear.
- If you finish early, check all of your work, then bring this packet up to the front of the class.
- Good luck!

# Section A.

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INSTRUCTIONS:

1. Each problem in this section is worth **one point** and no partial credit will be given.
2. You do not need to show your work or provide justification on any problem in Section A.
3. Make sure to **clearly box or circle your answer**.

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A1. Determine which of the sets below are vector subspaces of $\mathbb{R}^2$. **Clearly circle** your selection for each part. Note that you do not need to show any work.

a) The set of vectors satisfying the matrix-vector equation

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

**is a vector subspace** $\boxed{\textbf{is not a vector subspace}}$

b) The set of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ so that $x + y = 0$.

$\boxed{\textbf{is a vector subspace}}$ **is not a vector subspace**

c) The set of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ so that $x, y \geq 0$.

**is a vector subspace** $\boxed{\textbf{is not a vector subspace}}$

A2. Find a nontrivial solution to the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are defined below, or state that the vectors are linearly independent

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

**Solution.** $(x_1, x_2, x_3) = (1, -1, 1)$

A3. Find a basis for the subspace

$$V = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right)$$

**Solution.** $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}\right\}$

# Section B.

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Instructions:

1. Each problem in this section is worth **5 points**.
2. You must provide justification for all of your answers in Section B.
3. Points will be awarded based on the rubric below. Note that half points may be awarded, and further rubric items may be added to cover potential cases not outlined below.

| **Points** | **Rubric** |
|---|---|
| 5 | Solution is presented with clear justification that is logically complete and correct. May include minor typos and computational errors if they do not majorly impact the argument. No important steps are missing or assumed. All assumptions and special cases have been covered. All suggestions for improvement come under the category of "improvements for clarity" rather than "correcting logical errors". Omission of details will be judged depending on context of the material, with simpler steps being acceptable for omission when covering more advanced topics. |
| 4 | Solution is close to full and complete, but contains either a computational error or an error in reasoning that majorly impacts the argument. This score is also appropriate for solutions that are mathematically sound but confusingly written. |
| 3 | Solution is incorrect, but understanding of the problem was demonstrated and student provided a clear outline of a potential approach with information about where they got stuck **-or-** solution is correct but justification is insufficient or so confusingly written that it cannot be followed with a reasonable amount of effort. |
| 2 | Solution is incorrect, but student demonstrated understanding of the problem **-or-** solution is correct and student did not provide justification for their answer. |
| 1 | Solution is incorrect and student did not demonstrate understanding of the problem, but did demonstrate some knowledge of relevant material. |
| 0 | Solution is incorrect or incomplete, and there was no demonstration of knowledge of relevant material. |

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B1. True or False: if $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly *dependent* vectors, then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are also linearly dependent. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample.**

**Solution.** This is false. For example, the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

are linearly dependent because

$$\vec{v}_1 + \vec{v}_2 + \vec{v}_3 - \vec{v}_4 = \vec{0}$$

but $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

B2. Show that if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then so is $\{\vec{v}_1 + \vec{v}_2 + \vec{v}_3, \vec{v}_2, \vec{v}_3\}$.

**Solution.** This is true. Suppose that

$$x_1(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}.$$

Rewriting the left hand side gives

$$x_1\vec{v}_1 + (x_1 + x_2)\vec{v}_2 + (x_1 + x_3)\vec{v}_3 = \vec{0}.$$

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, we must have $x_1 = 0, x_1 + x_2 = 0, x_1 + x_3 = 0$. This gives $x_1 = x_2 = x_3 = 0$ as needed.

B3. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in $\mathbb{R}^n$. Show that if $m > n$ the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is not a basis for $\mathbb{R}^n$.

**Solution.** Let $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{pmatrix}$ and note that $A$ is an $n \times m$ matrix. Since $m > n$, $A$ has more columns than rows, and so the reduced row echelon form of $A$ has a column without a pivot. Hence, the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly dependent and so they cannot form a basis for $\mathbb{R}^n$.