EXAM PREP 4

Name: ______________________________

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General Instructions:

- This exam prep contains two sections: Section A includes computational problems, and Section B includes conceptual and proof-based problems. Please read the instructions at the beginning of each section carefully.
- You are allowed a full page of notes (front and back).
- No calculators or any other electronics are permitted.
- Write your answers clearly and make sure your handwriting is legible. If we cannot read your work, we will not grade the problem.
- All work must be completed in the space provided. If you need scratch paper, there is some at the front of the class. Please note that scratch work will not be graded.
- Please ask questions if anything is unclear.
- If you finish early, check all of your work, then bring this packet up to the front of the class.
- Good luck!

Section A.

INSTRUCTIONS:

1. Each problem in this section is worth **one point** and no partial credit will be given.
2. You do not need to show your work or provide justification on any problem in Section A.
3. Make sure to **clearly box or circle your answer**.

A1. Let $\mathcal{B}$ be the basis for $\mathbb{R}^3$ defined by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Find $[\vec{x}]_{\mathcal{B}}$, where $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. That is, find the coordinates for $\vec{x}$ with respect to the basis $\mathcal{B}$.

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}.$$

A2. Find a basis for Nul(A) where

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

Note that there are multiple correct answers. Here's one:

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

A3. Determine which of the functions below are linear. **Clearly circle** your selection for each part. Note that you do not need to show any work.

a) $F : \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + yz \\ z \end{pmatrix}$.

is linear $\quad$ $\boxed{\textbf{is not linear}}$

b) $G : \mathbb{R}^n \to \mathbb{R}^n$ defined by $G(\vec{x}) = \vec{0}$

$\boxed{\textbf{is linear}}$ $\quad$ **is not linear**

c) $H : \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{x} \mapsto \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

is linear $\quad$ $\boxed{\textbf{is not linear}}$

A4. Find the defining matrix of the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ which first rotates every vector counterclockwise by an angle of $270°$ and then stretches each vector by a factor of 5 (that is, makes each vector 5 times longer). Note that you may assume F is linear.

We have $F = T_A$ where

$$A = \begin{pmatrix} 0 & -5 \\ -5 & 0 \end{pmatrix}$$

Section B.

INSTRUCTIONS:

1. Each problem in this section is worth **5 points**.
2. You must provide justification for all of your answers in Section B.
3. Points will be awarded based on the rubric below. Note that half points may be awarded, and further rubric items may be added to cover potential cases not outlined below.

Points	**Rubric**
5	Solution is presented with clear justification that is logically complete and correct. May include minor typos and computational errors if they do not majorly impact the argument. No important steps are missing or assumed. All assumptions and special cases have been covered. All suggestions for improvement come under the category of "improvements for clarity" rather than "correcting logical errors". Omission of details will be judged depending on context of the material, with simpler steps being acceptable for omission when covering more advanced topics.
4	Solution is close to full and complete, but contains either a computational error or an error in reasoning that majorly impacts the argument. This score is also appropriate for solutions that are mathematically sound but confusingly written.
3	Solution is incorrect, but understanding of the problem was demonstrated and student provided a clear outline of a potential approach with information about where they got stuck **-or-** solution is correct but justification is insufficient or so confusingly written that it cannot be followed with a reasonable amount of effort.
2	Solution is incorrect, but student demonstrated understanding of the problem **-or-** solution is correct and student did not provide justification for their answer.
1	Solution is incorrect and student did not demonstrate understanding of the problem, but did demonstrate some knowledge of relevant material.
0	Solution is incorrect or incomplete, and there was no demonstration of knowledge of relevant material.

B1. Suppose that V and W are both vector subspaces of $\mathbb{R}^n$. Let $V+W$ be the subset of $\mathbb{R}^n$ defined by

$$V+W = \{\vec{v}+\vec{w} \mid \vec{v} \in V \text{ and } \vec{w} \in W\}.$$

Show that $V+W$ is a vector subspace of $\mathbb{R}^n$.

Proof. Since $\vec{0}$ is an element of V and W, then

$$\vec{0} = \vec{0}+\vec{0} \in V+W.$$

Next, take $\vec{v}+\vec{w} \in V+W$ and a constant $c \in \mathbb{R}$. Then we have

$$c(\vec{v}+\vec{w}) = c\vec{v}+c\vec{w}$$

and since $c\vec{v} \in V$ and $c\vec{w} \in W$ then we have

$$c(\vec{v}+\vec{w}) \in V+W.$$

Finally, take $\vec{v}_1+\vec{w}_1, \vec{v}_2+\vec{w}_2 \in V+W$. Then we have

$$(\vec{v}_1+\vec{w}_1)+(\vec{v}_2+\vec{w}_2) = (\vec{v}_1+\vec{v}_2)+(\vec{w}_1+\vec{w}_2),$$

and since $\vec{v}_1+\vec{v}_2 \in V$ and $\vec{w}_1+\vec{w}_2 \in W$ then we have $(\vec{v}_1+\vec{w}_1)+(\vec{v}_2+\vec{w}_2) \in V+W$. Hence, $V+W$ is a vector subspace of $\mathbb{R}^n$.

□

B2. True or False: If V and W are vector subspaces of $\mathbb{R}^n$ then $\dim(V+W) = \dim(V)+\dim(W)$. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample.**

This is false. For example, let V and W be any vector subspaces with $V = W$. Observe that $V+W = V$, and so $\dim(V+W) = \dim(V)$. (Note that this claim would be true if we also knew that $V \cap W = \{\vec{0}\}$).

B3. True or False: If A is an $n \times m$ matrix with $n < m$, then $\text{nullity}(A) > 0$. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample.**

This is true. Since $n < m$, we know that A must have a column with no pivot. So, the matrix-vector equation

$$A\vec{x} = \vec{0}$$

has a nontrivial solution, which gives $\text{Nul}(A) \neq \{\vec{0}\}$ and so $\text{nullity}(A) = \dim(\text{Nul}(A)) > 0$.