Name: ______________________________

Andrew ID: ________________________

GENERAL INSTRUCTIONS:

- This exam prep contains two sections: Section A includes computational problems, and Section B includes conceptual and proof-based problems. Please read the instructions at the beginning of each section carefully.
- You are allowed a full page of notes (front and back).
- No calculators or any other electronics are permitted.
- Write your answers clearly and make sure your handwriting is legible. If we cannot read your work, we will not grade the problem.
- All work must be completed in the space provided. If you need scratch paper, there is some at the front of the class. Please note that scratch work will not be graded.
- Please ask questions if anything is unclear.
- If you finish early, check all of your work, then bring this packet up to the front of the class.
- Good luck!

Section A.

INSTRUCTIONS:

1. Each problem in this section is worth **one point** and no partial credit will be given.
2. You do not need to show your work or provide justification on any problem in Section A.
3. Make sure to **clearly box or circle your answer**.

A1. Compute the matrix product

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 5 & 10 \\ 3 & 6 \end{pmatrix}$$

A2. Find the inverse of the matrix below or state that the inverse does not exist.

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -2 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix}$$

A3. Determine which of the following linear transformations are injective and which are surjective. **Clearly circle** your selection for each part. Circle both if the transformation is bijective. Note that you do not need to show any work.

a) $F : \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x-y \\ 2y \end{pmatrix}$.

is injective **is surjective**

b) $G : \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x-2z \\ y+z \\ x+y-z \end{pmatrix}$

is injective **is surjective**

c) $H : \mathbb{R}^n \to \mathbb{R}^m$, $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$ where $n > m$

is injective **is surjective**

Section B.

INSTRUCTIONS:

1. Each problem in this section is worth **5 points**.
2. You must provide justification for all of your answers in Section B.
3. Points will be awarded based on the rubric below. Note that half points may be awarded, and further rubric items may be added to cover potential cases not outlined below.

Points	**Rubric**
5	Solution is presented with clear justification that is logically complete and correct. May include minor typos and computational errors if they do not majorly impact the argument. No important steps are missing or assumed. All assumptions and special cases have been covered. All suggestions for improvement come under the category of "improvements for clarity" rather than "correcting logical errors". Omission of details will be judged depending on context of the material, with simpler steps being acceptable for omission when covering more advanced topics.
4	Solution is close to full and complete, but contains either a computational error or an error in reasoning that majorly impacts the argument. This score is also appropriate for solutions that are mathematically sound but confusingly written.
3	Solution is incorrect, but understanding of the problem was demonstrated and student provided a clear outline of a potential approach with information about where they got stuck **-or-** solution is correct but justification is insufficient or so confusingly written that it cannot be followed with a reasonable amount of effort.
2	Solution is incorrect, but student demonstrated understanding of the problem **-or-** solution is correct and student did not provide justification for their answer.
1	Solution is incorrect and student did not demonstrate understanding of the problem, but did demonstrate some knowledge of relevant material.
0	Solution is incorrect or incomplete, and there was no demonstration of knowledge of relevant material.

B1. True or False: if A and B are invertible $n \times n$ matrices, then $A + B$ is also an invertible matrix. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample**.

This is false. For example, let $A = I_n$ and $B = -I_n$. Then A and B are invertible, but $A + B$ is the zero matrix, which is not invertible (because, for example, the RREF of the zero matrix is the zero matrix).

B2. True or False: For any $n \times m$ matrix A, $\text{nullity}(A) = \text{nullity}(A^\top)$. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample**.

Solution. This is false. For example, let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then, A has a pivot in every column and so $\text{nullity}(A) = 0$. But

$$A^\top = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

only has two pivots and three columns, so by rank-nullity we have $\text{nullity}(A) = 1$.

B3. True or False: If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is an injective linear transformation, then $\{F(\vec{v}_1), \ldots, F(\vec{v}_n)\}$ is also a basis for $\mathbb{R}^n$. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample**.

This is true. Suppose that

$$c_1 F(\vec{v}_1) + c_2 F(\vec{v}_2) + \cdots + c_n F(\vec{v}_n) = \vec{0}$$

for some $c_i \in \mathbb{R}$. Since F is linear, this gives

$$F(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = \vec{0}. \tag{1.1}$$

So, $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ is in $\ker F$. But since F is injective we must have $\ker F = \{\vec{0}\}$ and so

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}.$$

But since the $\vec{v}_i$ are linearly independent, this means that

$$c_1 = c_2 = \cdots = c_n = 0$$

is the only solution to (1.1). Hence, the set $\{F(\vec{v}_1), \ldots, F(\vec{v}_n)\}$ is linearly independent. Since any set of n linearly independent vectors spans $\mathbb{R}^n$, this set is a basis for $\mathbb{R}^n$.