

Name: _____

Andrew ID: _____

GENERAL INSTRUCTIONS:

- This exam prep contains two sections: Section A includes computational problems, and Section B includes conceptual and proof-based problems. Please read the instructions at the beginning of each section carefully.
- You are allowed a full page of notes (front and back).
- No calculators or any other electronics are permitted.
- Write your answers clearly and make sure your handwriting is legible. If we cannot read your work, we will not grade the problem.
- All work must be completed in the space provided. If you need scratch paper, there is some at the front of the class. Please note that scratch work will not be graded.
- Please ask questions if anything is unclear.
- If you finish early, check all of your work, then bring this packet up to the front of the class.
- Good luck!

Section A.

INSTRUCTIONS:

1. Each problem in this section is worth **one point** and no partial credit will be given.
 2. You do not need to show your work or provide justification on any problem in Section A.
 3. Make sure to **clearly box or circle your answer**.
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A1. Find the determinant of the following matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 2 & 1 \end{pmatrix}$$

$$\boxed{\det(A) = 6}$$

A2. Find all values of k so that the following matrix is invertible

$$A = \begin{pmatrix} 1 & 5 & 2 \\ 0 & k & k^2 \\ 1 & 1 & -1 \end{pmatrix}.$$

Using the determinant, we see that A is invertible for all real numbers k **except** for $k = 0$ and $k = 3/4$.

A3. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 5 & -1 & 4 \\ 0 & 2 & -1 & 3 & 2 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

(and then don't panic, think about what makes these computations simpler). For any **one** eigenvalue λ of A , find the dimension of the λ -eigenspace E_λ .

The eigenvalues of A are $\lambda = 0, 1$ and 2 . Here's a worked out solution for $\lambda = 1$ (note that the other computations are similar). Recall that $E_1 = \text{Nul}(A - I_5)$. We have

$$A - \lambda I_5 = \begin{pmatrix} 0 & 3 & 5 & -1 & 4 \\ 0 & 1 & -1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is a matrix in row echelon form with three pivots, and so $\text{rank}(A - I_5) = 3$. So, by rank-nullity we have $\boxed{\dim E_1 = 2}$. Similarly, we get $\boxed{\dim E_0 = 1}$ and $\boxed{\dim E_2 = 2}$.

Section B.

INSTRUCTIONS:

1. Each problem in this section is worth **5 points**.
2. You must provide justification for all of your answers in Section B.
3. Points will be awarded based on the rubric below. Note that half points may be awarded, and further rubric items may be added to cover potential cases not outlined below.

Points	Rubric
5	Solution is presented with clear justification that is logically complete and correct. May include minor typos and computational errors if they do not majorly impact the argument. No important steps are missing or assumed. All assumptions and special cases have been covered. All suggestions for improvement come under the category of “improvements for clarity” rather than “correcting logical errors”. Omission of details will be judged depending on context of the material, with simpler steps being acceptable for omission when covering more advanced topics.
4	Solution is close to full and complete, but contains either a computational error or an error in reasoning that majorly impacts the argument. This score is also appropriate for solutions that are mathematically sound but confusingly written.
3	Solution is incorrect, but understanding of the problem was demonstrated and student provided a clear outline of a potential approach with information about where they got stuck -or- solution is correct but justification is insufficient or so confusingly written that it cannot be followed with a reasonable amount of effort.
2	Solution is incorrect, but student demonstrated understanding of the problem -or- solution is correct and student did not provide justification for their answer.
1	Solution is incorrect and student did not demonstrate understanding of the problem, but did demonstrate some knowledge of relevant material.
0	Solution is incorrect or incomplete, and there was no demonstration of knowledge of relevant material.

- B1. True or False: If A is an $n \times n$ matrix with $\text{Nul}(A) = \{\vec{0}\}$ then $\det(A) \neq 0$. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample.**

This is true. By assumption, $\text{nullity}(A) = 0$. So, by the Rank-Nullity Theorem we have $\text{rank}(A) = n$. This implies that A is invertible, and so $\det(A) \neq 0$.

- B2. A matrix A is called SKEW-SYMMETRIC if $A = -A^T$. Show that when n is odd, any $n \times n$ skew-symmetric matrix is not invertible.

We have $\det(A) = \det(-A^T) = (-1)^n \det(A)$. Since n is odd we know that $(-1)^n = -1$ and so

$$\det(A) = -\det(A) \Rightarrow \det(A) = 0.$$

So, A is not invertible.

- B3. True or False: If the only eigenvalue of an $n \times n$ matrix A is 0, then A must be equal to the zero matrix. **If true, provide a proof. If false, provide a counterexample and justify why this is a counterexample.**

This is false. For example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

only has eigenvalues equal to 0 but is not equal to the zero matrix.