4. Suppose $x, y \in \mathbb{R}$ satisfy the system:

$$\begin{cases} ax + by = r & (1) \\ cx + dy = s & (2) \end{cases}$$

Multiply (1) by c and (2) by a:

$$\begin{cases} cax + cby = cr & (3) \\ acx + ady = as & (4) \end{cases}$$

Subtract (4) − (3) to get:

$$(ad - bc)y = as - cr \quad (5)$$

Since $ad - bc \neq 0$, we can divide (5) by $ad - bc$ to get:

$$y = \frac{as - cr}{ad - bc}$$

On the other hand, if we multiply (1) by d and (2) by b:

$$\begin{cases} dax + dby = dr & (5) \\ bcx + bdy = bs & (6) \end{cases}$$

Subtract (5) $-$ (6):

$$(ad - bc)\,x = dr - bs$$

Since $ad - bc \neq 0$, we can divide by $ad - bc$:

$$x = \frac{dr - bs}{ad - bc}$$

This shows that the only solution is

$$\left(\frac{as - cr}{ad - bc}, \frac{dr - bs}{ad - bc} \right)$$

5. There are 4 possibilities for the number of pivots, namely 0, 1, 2 or 3.

<u>0 pivots</u>:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>1 pivot</u>:

$$\begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>2 pivots</u>:

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

3 pivots:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8.

	No Solutions	One Solution	Infinitely Many Solutions
$n=2, m=2$	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$	$\left(\begin{array}{cc\|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$
$n=2, m=3$	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$	(*)	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$
$n=3, m=2$	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$
$n=3, m=3$	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

(*): This is impossible. For the system to have a unique sol, the non-augmented matrix must have a pivot in every column. But the number of pivots is at most the number of rows, which is 2, which is less than the number of columns (3).

9. True. Let n be the number of equations and m be the number of variables, so $n < m$.

Let A be the reduced row echelon form of the augmented matrix of the system, so A is an $n \times (m+1)$ matrix.

A unique solution exists if and only if there is a pivot in each of the first m columns of A (see theorem 1.31 in the lecture notes). But the number of pivots it at most n, which is $< m$, so at least one of the first m columns has no pivots.

10. False. Consider the system whose augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

This has 2 variables, 3 equations and a unique solution $(0, 0)$.

11. True. Let i be the row whose pivot occurs in column 3, so i has the form $(0 \quad 0 \quad d)$, where $d \in \mathbb{R}$, $d \neq 0$, (because the augmented matrix has 3 columns). This implies that the system is inconsistent (it corresponds to the equation $0x + 0y = d \neq 0$

13. False: Consider a system with matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

This has a row of zeros, but is inconsistent.

14. False, consider a system with matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This matrix is in row echelon form, has a row of zeros, and yet it is inconsistent.