5. True. $\vec{v} = (-1)\cdot\vec{u} + (-1)\cdot(\vec{u}-\vec{v})$ is a linear combination of $\vec{u}$ and $\vec{u}-\vec{v}$, hence $\vec{v} \in \text{span}(\vec{u}, \vec{u}-\vec{v})$.

6. False. Let $\vec{u} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\vec{v} = \begin{pmatrix}2\\0\end{pmatrix}$. Then $\begin{pmatrix}0\\1\end{pmatrix} \notin \text{span}(\vec{u}, \vec{v})$. Suppose otherwise, then $\exists\, \alpha, \beta \in \mathbb{R} \left(\begin{pmatrix}0\\1\end{pmatrix} = \alpha\vec{u} + \beta\vec{v} \right)$

$$\Rightarrow \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha + 2\beta\\0\end{pmatrix} \Rightarrow 1 = 0, \text{ contradiction.}$$

So, $\begin{pmatrix}0\\1\end{pmatrix} \notin \text{span}(\vec{u}, \vec{v})$

7. True. We show that $\text{span}(\vec{u}, \vec{u}+\vec{v}) = \mathbb{R}^2$.

$\subseteq$) If $\vec{w} \in \text{span}(\vec{u}, \vec{u}+\vec{v})$, then $\exists\, \alpha, \beta \in \mathbb{R} \left(\vec{w} = \alpha \underbrace{\vec{u}}_{\text{in } \mathbb{R}^2} + \beta \underbrace{(\vec{u}+\vec{v})}_{\text{in } \mathbb{R}^2} \right)$. So, $\vec{w} \in \mathbb{R}^2$.

$\supseteq$) Let $\vec{w} \in \mathbb{R}^2$. We know that $\mathbb{R}^2 = \text{span}(\vec{u}, \vec{v})$, so $\exists\, \alpha, \beta \in \mathbb{R} \left(\vec{w} = \alpha\vec{u} + \beta\vec{v} \right.$

Then $\vec{w} = \alpha\vec{u} + \beta\vec{v} = \alpha\vec{u} - \beta\vec{u} + \beta\vec{u} + \beta\vec{v}$

$= \underbrace{(\alpha - \beta)}_{\in \mathbb{R}}\vec{u} + \beta(\vec{u} + \vec{v}) \implies \vec{w} \in \operatorname{span}(\vec{u}, \vec{u}+\vec{v})$

8. False. We give two arguments.

pf 1: Take any $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Then

$\dim\left(\operatorname{span}(\vec{u}+\vec{v}, \vec{v}+\vec{w})\right) \leq 2 < 3 = \dim(\mathbb{R}^3)$

$\implies \operatorname{span}(\vec{u}+\vec{v}, \vec{v}+\vec{w}) \neq \mathbb{R}^3.$

pf 2: Let $\vec{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{w} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\implies \vec{u} + \vec{v} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\vec{v} + \vec{w} = \begin{pmatrix}0\\1\\1\end{pmatrix}$. We

show that $\operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\} \neq \mathbb{R}^3$ by

showing that $\begin{pmatrix}0\\0\\1\end{pmatrix} \notin \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$.

Suppose towards a contradiction that

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\} \Rightarrow \exists \alpha, \beta \in \mathbb{R} \left(\begin{pmatrix}0\\0\\1\end{pmatrix} = \alpha\begin{pmatrix}1\\1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\\1\end{pmatrix}\right)$$

$$\Rightarrow \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\ \alpha+\beta\\ \beta\end{pmatrix} \Rightarrow \begin{cases} 0 = \alpha \\ 0 = \alpha + \beta \\ 1 = \beta \end{cases}$$

But then $0 = \alpha + \beta = 0 + 1 = 1$, contradiction.

9. We can represent each solution by a vector, so we have:

$\vec{v}_1 = (36, 50, 14)$, $\vec{v}_2 = (44, 46, 10)$, $\vec{v}_3 = (38, 49, 13)$

The question of a solution being constructible from the other 2 is equivalent to the three vectors being linearly dependent.

Letting $A = \begin{pmatrix}\vec{v}_1\\ \vec{v}_2\\ \vec{v}_3\end{pmatrix}$ be the matrix with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3$, we want to determine whether the RREF of A has a pivot in every row.

Row reducing:

$$\begin{pmatrix} 36 & 50 & 14 \\ 44 & 46 & 10 \\ 38 & 49 & 13 \end{pmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 36 & 50 & 14 \\ 8 & -4 & -4 \\ 2 & -1 & -1 \end{pmatrix} \xrightarrow{R_3 - \frac{1}{4}R_2} \begin{pmatrix} 36 & 50 & 14 \\ 4 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

This shows that $\vec{v_3}$ is a linear combination of $\vec{v_1}$ and $\vec{v_2}$. To find the exact linear combination,

$$\vec{0} = (\vec{v_3} - \vec{v_1}) - \frac{1}{4}(\vec{v_2} - \vec{v_1})$$

$$\Longrightarrow \quad \vec{v_3} = \frac{3}{4}\vec{v_1} + \frac{1}{4}\vec{v_2}$$

0. (a) $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

(b) $$\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

These 3

(c) Not possible. Let $\vec{v_1}, \vec{v_2}, \vec{v_3} \in \mathbb{R}^2$. Then $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ are lin dep $\iff \exists \vec{x} \neq \vec{0} \ (A\vec{x} = \vec{0})$, where $\vec{A} = (\vec{v_1} \mid \vec{v_2} \mid \vec{v_3})$. Since A has 2 rows, the RREF of A has ≤ 2 pivots. Since A has 3 columns, not every col of the RREF of A has a pivot, hence the homogeneous system $A\vec{x} = \vec{0}$ has a non-trivial solution.

(d) Not possible. The proof is the same as (c), with a 3×4 matrix instead of 2×3.

(e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(f) Not possible, see the argument for 10(d).

12. False, consider $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. None is a scalar mult of another, but $\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$.

13. True. Let $\alpha, \beta, \gamma \in \mathbb{R}$ be s.t.

$\vec{0} = \alpha \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3$. Then

$\vec{0} = \alpha \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3 + 0 \cdot \vec{v}_4$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are lin. ind,

$\alpha = 0, \ \beta = 0, \ \gamma = 0.$

So $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are lin. ind.