1. Let $\vec{v}_1, \ldots, \vec{v}_n$ be linearly independent in $\mathbb{R}^n$. Fix $\vec{b} \in \mathbb{R}^n$ and let

$A = (\vec{v}_1 | \cdots | \vec{v}_n)$. By linear indep., the RREF form of A has a pivot in every column (because the system $A\vec{x} = \vec{0}$ has only the trivial solution).

As A is $n \times n$, the RREF form of A must have a pivot in every row, hence $A\vec{x} = \vec{b}$ is consistent. Fix $\vec{\alpha} \in \mathbb{R}^n$ st $A\vec{\alpha} = \vec{b}$, say $\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Then

$$\vec{b} = \sum_{k=1}^{n} \alpha_k \vec{v}_k .$$

This shows that $\vec{v}_1, \ldots, \vec{v}_k$ span $\mathbb{R}^n$.

2. (a) Yes.

- $\vec{0} = \begin{pmatrix} 2 \cdot 0 + 0 \\ 0 - 0 \\ 0 + 0 \end{pmatrix} \in W$

- Let $\vec{v}, \vec{w} \in W \Rightarrow \exists\, x, y \in \mathbb{R}$ $\vec{v} = \begin{pmatrix} 2x + y \\ x - y \\ x + y \end{pmatrix}$

and $\exists\, \alpha, \beta \in \mathbb{R}$ $\vec{w} = \begin{pmatrix} 2\alpha + \beta \\ \alpha - \beta \\ \alpha + \beta \end{pmatrix}$.

$$\Longrightarrow \vec{v} + \vec{w} = \begin{pmatrix} 2(x+\alpha) + (y+\beta) \\ x + \alpha - (y+\beta) \\ x + \alpha + y + \beta \end{pmatrix} \in W$$

- Let $\vec{v} \in W,\ \lambda \in \mathbb{R} \Rightarrow \exists\, x, y \in \mathbb{R}$ $\vec{v} = \begin{pmatrix} 2x + y \\ x - y \\ x + y \end{pmatrix}$

$$\Rightarrow \lambda \vec{v} = \begin{pmatrix} 2(\lambda x) + \lambda y \\ \lambda x - \lambda y \\ \lambda x + \lambda y \end{pmatrix} \in W$$

(b) W is not a subspace, because $\vec{0} \notin W$.

Suppose towards a contradiction that $\vec{0} \in W$

$$\Rightarrow \exists x \in \mathbb{R} \begin{pmatrix} x+1 \\ x-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+1=0 \\ x-1=0 \end{cases}$$

$$\Rightarrow -1 = x = 1, \quad \text{contradiction.}$$

(c) W is not a subspace of $\mathbb{R}^3$, because it is not closed under scalar multiplicat.

Indeed, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 \\ 1\cdot 1 \\ 1\cdot 1 \end{pmatrix} \in W$ but $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \notin W$.

Otherwise, $\exists x,y,z \in \mathbb{R}$ $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} xy \\ yz \\ yz \end{pmatrix}$. Then

x and y must have $\neq$ signs (as $xy < 0$), wlog $x < 0 < y$. Then, as $yz = -1 < 0$, $z < 0 < y$. But then $x \cdot z > 0$, contradicting $xz = -1$.

4. • As W and V are subspaces, $\vec{0} \in W$ and $\vec{0} \in V$, $\vec{0} \in W \cap V$.

• Let $\vec{x}, \vec{y} \in W \cap V$, then

$$\begin{cases} \vec{x} \in W \wedge \vec{y} \in W \longrightarrow \vec{x} + \vec{y} \in W & \text{(because } W \text{ is a subspace)} \\ \vec{x} \in V \wedge \vec{y} \in V \longrightarrow \vec{x} + \vec{y} \in V & \text{(because } V \text{ is a subspace)} \end{cases}$$

$$\Longrightarrow \vec{x} + \vec{y} \in W \cap V$$

• Let $\vec{x} \in W \cap V$, $\alpha \in \mathbb{R}$, then

$$\begin{cases} \alpha \vec{x} \in W & \text{(because } W \text{ is a subspace)} \\ \alpha \vec{x} \in V & \text{(" } V \text{ " " ")} \end{cases}$$

$$\Longrightarrow \alpha \vec{x} \in W \cap V$$

$\therefore$ $W \cap V$ is a subspace of $\mathbb{R}^n$.

5. Let $n=2$, $W = \text{span}\begin{pmatrix}1\\0\end{pmatrix}$, $V = \text{span}\begin{pmatrix}0\\1\end{pmatrix}$

$\Rightarrow$ $V \cup W$ is not a subspace of $\mathbb{R}^2$,

because $\begin{pmatrix}1\\0\end{pmatrix} \in V \cup W$, $\begin{pmatrix}0\\1\end{pmatrix} \in V \cup W$, but

$\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \notin V \cup W.$

6. Say $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_m\}$. Let $v \in \mathbb{R}^n \setminus V$.

Suppose $\alpha_1, \dots, \alpha_{m+1} \in \mathbb{R}$ are s.t.

$$\vec{0} = \alpha_1 \vec{b}_1 + \dots + \alpha_m \vec{b}_m + \alpha_{m+1} \vec{v} \qquad (*)$$

If $\alpha_{m+1} \neq 0$, then by $(*)$

$$\vec{v} = -\frac{\alpha_1}{\alpha_{m+1}} \vec{b}_1 - \dots - \frac{\alpha_m}{\alpha_{m+1}} \vec{b}_m$$

$\Rightarrow$ $\vec{v} \in V$ because V is closed under linear combinations and $\mathcal{B} \subseteq V$.

This contradicts $\vec{v} \notin V$. So, $\alpha_{m+1} = 0$ and $(*)$ becomes

$$\vec{0} = \alpha_1 \vec{b}_1 + \cdots + \alpha_m \vec{b}_m$$

As $\mathcal{B}$ is a basis, it is lin. ind, hence $\alpha_i = 0$ $\forall i \in \{1, \ldots, n\}$.

7. Recall that, if $W \subseteq \mathbb{R}^n$ is a subspace, with $\dim(W) < n$, then $W \neq \mathbb{R}^n$.

If $m = n$, then $\mathcal{B}$ is a basis for $\mathbb{R}^n$ by ex 1.

Recursively define $\vec{b}_{m+1}, \ldots, \vec{b}_n$ as follows: at stage k, $m+1 \leq k \leq n$,

if $\{\vec{b}_i \mid 1 \leq i < k\}$ is linearly independent,

let $\vec{b}_k \in \mathbb{R}^n \setminus \operatorname{span}\{\vec{b}_1, \ldots, b_{k-1}\}$.

By Exercise 6, $\{\vec{b}_1, \ldots, \vec{b}_{k+1}\}$ is linearly independent, and the construction can proceed.

At stage n, the recursion halts, yielding $\vec{b}_1, \ldots, \vec{b}_m, \vec{b}_{m+1}, \ldots, \vec{b}_n$ lin. ind. By ex 1, $C = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for $\mathbb{R}^n$. Certainly, $\mathcal{B} \subseteq C$.

8. Let $S = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2\}$, so that S is a subspace of $\mathbb{R}^5$ (see thm 3.3). We show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for S. By def, it spans S. To check linear indep, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ be st $\vec{0} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \beta_1 \vec{w}_1 + \beta_2 \vec{w}_2$.

$$\Rightarrow \quad \underbrace{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2}_{\substack{\cap \\ V}} = \underbrace{-\beta_1 \vec{w}_1 - \beta_2 \vec{w}_2}_{\substack{\cap \\ W}}$$

$\Rightarrow \quad \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 \in V \cap W = \{\vec{0}\}$, hence

$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = \vec{0}$, so $\alpha_1 = 0 = \alpha_2$

because $\vec{v}_1, \vec{v}_2$ is l. i.

By the same logic, $\beta_1 \vec{w}_1 + \beta_2 \vec{w}_2 = \vec{0}$,

hence $\beta_1 = 0 = \beta_2$.

$\Rightarrow \quad \{\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2\}$ lin. ind.

$\Rightarrow \quad \dim(S) = 4.$

12. By assumption, $\begin{pmatrix} -1 \\ 1 \\ c \end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\right\}$

$$\Rightarrow \exists \alpha, \beta \in \mathbb{R} \quad \begin{pmatrix} -1 \\ 1 \\ c \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \qquad (*)$$

$$\Rightarrow \begin{cases} \alpha + 2\beta = -1 \\ \alpha + \beta = 1 \end{cases} \Rightarrow \beta = -2 \Rightarrow \alpha = 3$$

$$\overset{(*)}{\Rightarrow} c = 3 \cdot 1 + (-2) \cdot 2 = -1$$

13. $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \in \operatorname{Null}\begin{pmatrix} 2 & 1 & a \\ 1 & 2 & b \end{pmatrix} \iff \begin{pmatrix} 2 & 1 & a \\ 1 & 2 & b \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\iff \begin{cases} 4 - 1 + a = 0 \\ 2 - 2 + b = 0 \end{cases} \iff a = -3, \; b = 0$$