6. True. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and 1-1. Then $\ker(T) = \{0\}$ so, by the rank-nullity thm,

$$n = \dim(\mathbb{R}^n) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$
$$= \dim(\operatorname{im}(T))$$

As $\operatorname{im}(T)$ is a subspace of $\mathbb{R}^n$ of the same dimension, $\operatorname{im}(T) = \mathbb{R}^n$. So, T is onto.

7. True. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be linear. By the rank-nullity thm,

$$2 = \dim(\mathbb{R}^2) = \underbrace{\dim(\ker(T))}_{\geqslant 0} + \dim(\operatorname{im}(T))$$

$\Longrightarrow \dim(\operatorname{im}(T)) \leqslant 2$. But $\dim(\mathbb{R}^3) = 3$,

so $\operatorname{im}(T) \neq \mathbb{R}^3$, so T is not onto.

8. False. Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then T is linear and, if $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, then $T\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, so T is onto.

9. True. If $T: \mathbb{R}^3 \to \mathbb{R}^2$ is linear, then by the rk-null thm $3 = \dim(\mathbb{R}^3) = \dim(\mathrm{Ker}(T)) + \dim(\underbrace{\mathrm{im}(T)}_{\subseteq \mathbb{R}^2})$

$\leqslant \dim(\mathrm{Ker}(T)) + 2$

$\Longrightarrow \dim(\mathrm{Ker}(T)) \geqslant 1 \Longrightarrow \mathrm{Ker}(T) \neq \{\vec{0}\}$

$\Longrightarrow T$ <u>not</u> 1-1.

10. False. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, which has l.i. columns.

As $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \notin \mathrm{im}(T_A)$, T_A not onto.

11. True. It is enough to argue that $\ker(T_A) = \{\vec{0}\}$. Fix $\vec{x} \in \ker(T_A)$, so

$$\vec{0} = T_A \vec{x} = A\vec{x} = \sum_{i=1}^{n} x_k \underbrace{\mathrm{col}_i(A)}_{i^{th} \text{ col of } A}, \quad \text{where } \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

By linear indep, $x_i = 0$ $\forall i \in \{1, \dots, n\}$, hence $\vec{x} = \vec{0}$.

14. $$A^2 = \begin{pmatrix} 0\cdot 0 + 0 \cdot 0 & 0 \cdot a + a \, 0 \\ 0 \cdot 0 + 0 \cdot 0 & 0 \cdot a + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

For B it suffices to argue that, if $T_B(\vec{x}) = B\vec{x}$, then $\underbrace{T_B \circ T_B \circ \dots \circ T_B}_{n} = 0$ for some n. Here 0 denotes the identically $\vec{0}$ map $\mathbb{R}^3 \to \mathbb{R}^3$.

Note: $T_B \vec{e}_1 = \vec{0}$, $T_B \vec{e}_2 = a\vec{e}_1$, $T_B e_3 = b\vec{e}_1 + c\vec{e}_2$

$\Rightarrow T_B(T_B(\vec{e}_2)) = a T_B \vec{e}_1 = \vec{0}$ and

$$T_B(T_B(\vec{e}_3)) = b\underbrace{T_B\vec{e}_1}_{}^{\vec{0}} + c\,T_B\vec{e}_2 = ca\vec{e}_1$$

$$\Rightarrow T_B(T_B(T_B(\vec{e}_3))) = ca\,T_B\vec{e}_1 = \vec{0}.$$

So $T_B \circ T_B \circ T_B(\vec{e}_i) = \vec{0}$ for $i = 1, 2, 3$,

so $T_B \circ T_B \circ T_B = 0$, so $B^3 = 0$.

15. Since A is invertible, $\exists B\ (AB = I = BA)$

As $A^2 = A \Rightarrow \underbrace{BA}_{I}A = BA \Rightarrow IA = I$

$\Rightarrow A = I$.

16. False. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

but $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

17. False, let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

18. False, let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $A^2 = I_2$.