Chapter 6

1. We deal with each type of elementary matrix separately:

   - First, suppose that E is the matrix corresponding to the row operation "add c times row i to row j". Then E is obtained from I by the same operation, so E has the form

$$E = \begin{array}{c} \begin{array}{cccccc} & & i & & j & \end{array} \\ \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \\ \vdots & \ddots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & \dots & c & 0 & 1 & \dots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ i \\ \\ j \\ \\ \end{array}$$

   Note also that, by proposition 6.8, $\det(E) = \det(I) = 1$.

   Transposing, we obtain:

$$E^T = \begin{array}{c} \begin{array}{cccccc} & & i & & j & \end{array} \\ \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \\ \vdots & \ddots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & c & \dots \\ 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & \dots & 0 & 0 & 1 & \dots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ i \\ \\ j \\ \\ \end{array}$$

   But then E^T is the elementary matrix corresponding to the operation "add c times row j to row i", hence by proposition 6.8 applied again, $\det(E^T) = \det(I) = 1$. In particular, $\det(E) = \det(E^T)$.

   - Suppose that E is the matrix corresponding to the row operation "interchange rows i and j". Then E is obtained from I by performing that same operation, so E has the form

$$E = \begin{array}{c} \begin{array}{cccccc} & & i & & j & \end{array} \\ \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \\ \vdots & \ddots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 1 & \dots \\ 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & \dots & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ i \\ \\ j \\ \\ \end{array}$$

   But then $E^T = E$, so in particular $\det(E^T) = \det(E)$.

   - Finally, suppose that E is the elementary matrix corresponding to the operation "multiply row i by c". Then E is obtained from I by performing that same operation, so E has the form

$$E = \begin{array}{cc} \begin{array}{cccccc} & & i & & j & \end{array} & \\ \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \\ \vdots & \ddots & \dots & \dots & \dots & \dots \\ 0 & \dots & c & \dots & 0 & \dots \\ 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & \dots & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} & \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \end{array}$$

Again, this gives $E^T = E$ hence in particular $\det(E^T) = \det(E)$.

2.

(a) We already saw that, if E corresponds to the row operation "add c times row i to row j", then $\det(E) = 1$. If instead E corresponds to the row operation "interchange rows i and j", then by Proposition 6.8 $\det(E) = -\det(I) = -1$. By the same proposition, if E corresponds to the row operation "multiply row i by c", then $\det(E) = c$.

(b) EB is the matrix obtained from B by performing the row operation corresponding to E, so this follows by proposition 6.8 together with the values obtained in (a).

(c) If A is invertible, then A can be row reduced to I, sothere exists $k \in \mathbb{N}$ and elementary matrices $E_1, \dots, E_k$ such that $E_1 \dots E_k A = I$, so $A = E_k^{-1} \cdots E_1^{-1}$. Applying (b) k many times,

$$1 = \det(I) = \det(E_1 \dots E_k A) = \det(E_1) \cdots \det(E_k) \det(A),$$

and therefore

$$\det(A) = (\det(E_1))^{-1} \cdots (\det(E_k))^{-1} = (\det(E_k))^{-1} \cdots (\det(E_1))^{-1}$$

because multiplication of numbers is commutative.

We now make the following auxiliary observation: if E is an elementary matrix, then

$$\det(E^{-1}) = (\det(E))^{-1}.$$

To see this, note that E^{-1} is again an elementary matrix of the same type. Then argue by cases using the values computed in (a).

Now applying (b) k many times:

$$\begin{aligned} \det(A)\det(B) &= (\det(E_k))^{-1} \cdots (\det(E_1))^{-1} \det(A) \\ &= \det(E_k^{-1}) \cdots \det(E_1^{-1}) \det(B) \\ &= \det(E_k^{-1} \cdots E_1^{-1} B) \\ &= \det(AB). \end{aligned}$$

(d) If A is not invertible, then $\det(A) = 0$, so it suffices to argue that $\det(AB) = 0$. To do that, we show that AB must not be invertible. But this can be seen as follows: if AB were invertible, then there would exist some matrix C with $(AB)C = I$, hence $A(BC) = I$, which yields that A itself is invertible, contradiction.

3. $AA^{-1} = I$, hence $I = \det(AA^{-1}) = \det(A)\det(A^{-1})$ and so $\det(A^{-1}) = \frac{1}{\det(A)}$.

4. False, let I be the 2×2 identity matrix, so $\det(I) = 1$ and $\det(-I) = 1$.

6. True. Suppose A and B are $n \times n$ matrices and AB is invertible, so there exists C such that $(AB)C = I$. Note that then $A(BC) = I$, so the column space of A is contained in the column space of I, which is all of $\mathbb{R}^n$. So, the column space of A is $\mathbb{R}^n$. But A is square, and so A must be invertible. The proof for B is analogous, or can be obtained by writing $B = A^{-1}(AB)$, which is a product of invertible matrices hence invertible.

1 Chapter 7

2. Since $(A - \lambda I)^T = A^T - \lambda I$, it follows that $\chi_{A_T} = \det(A^T - \lambda I) = \det((A - \lambda I)^T) = \det(A - \lambda I) = \chi_A$ (recall that $\det(B) = \det(B^T)$ for all square matrices B).

3. Let λ be an an eigenvalue of A, so there exists some *nonzero* x with $Ax = \lambda x$. Then $\lambda x = Ax = A^2 x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$. Since $x \neq \vec{0}$, we infer that $\lambda = \lambda^2$, so $\lambda \in \{0, 1\}$.

4. False, let
$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
and
$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
Both A and B have only 0 as an eigenvalue (because they are nilpotent). Note that $e_2 = (0, 1)$ is an eigenvector of A (by $Ae_2 = \vec{0}$) but not an eigenvector of B (because $Ae_2 = e_1$).

5. False, the matrix $2I$ has only the eigenvalue 2, whereas its RREF is I, which has only the eigenvalue 1.

6. False. The matrix
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
is invertible (its a counterclockwise rotation by $\pi/2$), hence $R = I$ has eigenvalue 1 (and every vector is an eigenvector). But A has no real eigenvalues, so no eigenvectors in $\mathbb{R}^2$.