WORKSHEET 2 SOLUTIONS

This Worksheet will collected at the end of your recitation section and will be graded on completeness. We will return your graded worksheet back to you during recitation next week.

Chapter 2. The Geometry of Systems of Linear Equations

Recall that a system of linear equations is called HOMOGENEOUS if the constant term in every equation is equal to zero. Observe that equations of this type can be written in matrix-vector form as $A\vec{x} = \vec{0}$ where A is the coefficient matrix of the linear system, and $\vec{0}$ denotes the zero vector.

1. Observe that any homogeneous system $A\vec{x} = \vec{0}$ has the "trivial" solution $\vec{x} = \vec{0}$.

   a) Give an example of a matrix A where the *only* solution to the homogeneous system $A\vec{x} = \vec{0}$ is the trivial solution.

   Solution. Let
   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
   Then the system $A\vec{x} = \vec{0}$ only has the solution $\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

   b) Given an example of a matrix A so that the homogeneous system $A\vec{x} = \vec{0}$ has a nontrivial solution.

   Solution. Let
   $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$
   Then, the system $A\vec{x} = \vec{0}$ has the nontrivial solution
   $$\vec{x} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

   c) What must be true about the matrix A in order to guarantee a nontrivial solution to the homogeneous system $A\vec{x} = \vec{0}$?

   Solution. Note that the system $A\vec{x} = \vec{0}$ either has exactly one solution or infinitely many solutions. So, for this system to have a nontrivial solution, it must have infinitely many solutions. By Theorem 1.19 in our course lecture notes, we know that the system $\left(A \mid \vec{0}\right)$ has infinitely many solutions precisely when $\boxed{A \text{ has a column without a pivot}}$.

2. Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \text{ and } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Write the solution set to the linear system $A\vec{x} = \vec{0}$ as $\mathrm{Span}(\vec{v})$ for some vector $\vec{v}$ in $\mathbb{R}^3$.

Solution. Observe that A is row equivalent to the matrix in reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}.$$

So, all solutions to the matrix-vector equation $A\vec{x} = \vec{0}$ satisfy the system of linear equations

$$x_1 + 3x_3 = 0$$

$$x_2 - x_3 = 0.$$

Setting $x_3 = t$ as our free variables gives

$$x_1 = -3t \text{ and } x_2 = t.$$

So, all solutions to $A\vec{x} = \vec{0}$ are given by

$$\vec{x} = \begin{pmatrix} -3t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}.$$

for $t \in \mathbb{R}$. That is, the solution set to $A\vec{x} = \vec{0}$ is equal to $\boxed{\mathrm{Span}\left(\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}\right)}$.

3. Let

$$A = \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \end{pmatrix}, \text{ and } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Write the solution set to the linear system $A\vec{x} = \vec{0}$ as $\text{Span}(\vec{v}, \vec{w})$ for some vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^4$.

Solution. Note that the matrix A is already in reduced row echelon form. So, all solutions to the matrix-vector equation $A\vec{x} = \vec{0}$ satisfy the system of linear equations

$$x_1 - 8x_3 - 7x_4 = 0$$

$$x_2 + 4x_3 + 3x_4 = 0.$$

Letting $x_3 = s, x_4 = t$ be our free variables, we get

$$x_1 = 8s + 7t$$

$$x_2 = -4s - 3t.$$

So all solutions to $A\vec{x} = \vec{0}$ are given by

$$\vec{x} = \begin{pmatrix} 8s + 7t \\ -4s - 3t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} t$$

for $s, t, \in \mathbb{R}$. That is, the solution set to $A\vec{x} = \vec{0}$ is equal to $\boxed{\text{Span}\left(\begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}\right)}$.

4. Based on the previous two problems, how do you think we can express the solution set to any homogeneous linear system? We'll discuss this formally later in the semester, for now just try to make a conjecture and we'll revisit your thoughts in a few weeks.

Solution. In a few weeks, we'll show that the solution set to any homogeneous system is of the form $\text{Span}(\vec{v}_1, \ldots, \vec{v}_k)$.

5. Let A and $\vec{x}$ be as in Problem 2. What is the solution set of

$$A\vec{x} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

for arbitrary $b_1, b_2 \in \mathbb{R}$. How is this related to your answer in Problem 2?

Solution. Solving as in Problem 2, we get all solutions are of the form

$$t\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2b_1 - b_2 \\ b_2 - b_1 \\ 0 \end{pmatrix},$$

for $t \in \mathbb{R}$.

6. Let A and $\vec{x}$ be as in Problem 3. What is the solution set of

$$A\vec{x} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

for arbitrary $b_1, b_2 \in \mathbb{R}$. How does this compare to your answer in Problem 3?

Solution. Solving as in Problem 3, we get all solutions are of the form

$$\begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{pmatrix}$$

for $s, t \in \mathbb{R}$.

7. (Optional) The Theorem below should generalize what you found in the previous two problems. Fill in the details omitted in the proof below.

   Theorem 1. Let V denote the solution set to the homogeneous system $A\vec{x} = \vec{0}$. Suppose that the nonhomogeneous system $A\vec{x} = \vec{b}$ is consistent with particular solution $\vec{p}$. Then the solution set to the system $A\vec{x} = \vec{b}$ is equal to $\{\vec{p} + \vec{v} \mid \vec{v} \in V\}$.

   Before we start, let's figure out what we need to show. Recall that two sets X and Y are EQUAL if every element of X is also an element of Y, and every element of Y is an element of X. So, two show that the solution *set* of the system $A\vec{x} = \vec{b}$ is equal to the *set* proposed above, we need to show two things:

   (i) Every solution to $A\vec{x} = \vec{b}$ is in the set $\{\vec{p} + \vec{v} \mid \vec{v} \in V\}$.

   (ii) Every element in $\{\vec{p} + \vec{v} \mid \vec{v} \in V\}$ is a solution to $A\vec{x} = \vec{b}$.

   Proof. Let $\vec{y}$ be any solution to the system $A\vec{x} = \vec{b}$. Then,

$$\begin{aligned} A(\vec{y} - \vec{p}) &= A\vec{y} - A\vec{p} \\ &= \vec{b} - \vec{b} \\ &= \vec{0}. \end{aligned}$$

   Hence, $\vec{y} - \vec{p}$ is a solution to $A\vec{x} = 0$ and so we can write $\vec{y} - \vec{p} = \vec{v}$ for some $\vec{v} \in V$. This gives

$$\vec{y} = \vec{v} + \vec{p}$$

   for $\vec{v} \in V$ as desired. Conversely, suppose that $\vec{v} \in V$. That is, $A\vec{v} = \vec{0}$. Then,

$$\begin{aligned} A(\vec{p} + \vec{v}) &= A\vec{p} + A\vec{v} \\ &= \vec{b} + \vec{0} \\ &= \vec{b}. \end{aligned}$$

   So, $\vec{p} + \vec{v}$ is a solution to $A\vec{x} = \vec{b}$. □