WORKSHEET 3 **NAME:**

This Worksheet will collected at the end of your recitation section and will be graded on completeness. We will return your graded worksheet back to you during recitation next week.

Chapter 3. Vector Spaces

P1. Let $V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ be a vector subspace of $\mathbb{R}^n$, and let

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{pmatrix}.$$

Suppose that the reduced row echelon form of A has a pivot in columns 1, 3 and 4, and no pivot in column 2.

a) Show that $\vec{v}_2$ is in $\text{Span}(\vec{v}_1, \vec{v}_3, \vec{v}_4)$ *(Hint: think about which variable is free in the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$).*

$$A \xrightarrow[\text{reduce}]{\text{Row}} \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x_1 + \alpha x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{cases}$$

$(\alpha \neq 0)$

So, x_2 is free in the original eqn

It follows that $\begin{pmatrix} -\alpha \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is a solution to $\vec{0} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4$

hence $\vec{v}_2 = \alpha\vec{v}_1 \in \text{span}(\vec{v}_1, \vec{v}_3, \vec{v}_4)$

b) Explain how we know that $\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$ is a linearly independent set.

From (a), we know that there is an $\alpha \neq 0$ such that

$$\vec{0} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 \iff \begin{cases} x_1 + \alpha x_2 = 0 \\ x_3 = 0 \\ x_4 = 0. \end{cases}$$

Suppose $x_1, x_3, x_4 \in \mathbb{R}$ and $x_1\vec{v}_1 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$. Then $x_1\vec{v}_1 + 0\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$, so $x_3 = 0 = x_4$ and $x_1 = -\alpha \cdot 0 = 0$.

c) Conclude that $\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$ is a basis for V.

As $\vec{v}_2 \in \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, it follows that $\text{span}(\vec{v}_1, \vec{v}_3, \vec{v}_4) = V$. As $\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$ is lin ind, it is a basis for V.

P2. Find a basis and the dimension of the following vector subspaces of $\mathbb{R}^3$.

a) $V = \text{Span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$.

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4$

$$\begin{pmatrix}1 & 2 & 1 & 0\\ 1 & 1 & 2 & 0\\ 0 & 1 & -1 & 1\end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix}1 & 2 & 1 & 0\\ 0 & -1 & -1 & 0\\ 0 & 1 & -1 & 1\end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix}1 & 2 & 1 & 0\\ 0 & -1 & -1 & 0\\ 0 & 0 & -2 & 1\end{pmatrix}$$

Pivots in columns 1, 2 and 3, hence only the 4^{th} variable is free in the associated homogeneous system. So, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is l.i. but $\vec{v}_4 \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for V.

b) $W = \text{Span}\left(\begin{pmatrix}1\\-2\\2\end{pmatrix}, \begin{pmatrix}2\\-3\\4\end{pmatrix}, \begin{pmatrix}0\\4\\0\end{pmatrix}, \begin{pmatrix}-1\\5\\-2\end{pmatrix}\right)$.

$$\begin{pmatrix}1 & 2 & 0 & -1\\ -2 & -3 & 4 & 5\\ 2 & 4 & 0 & -2\end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + 2R_1} \begin{pmatrix}1 & 2 & 0 & -1\\ 0 & -1 & 4 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Columns 1 and 2 have pivots, hence $\left\{\begin{pmatrix}1\\-2\\2\end{pmatrix}, \begin{pmatrix}2\\-3\\4\end{pmatrix}\right\}$ is a basis for W.

P3. Let $V = \text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ be a vector subspace of $\mathbb{R}^n$ and let

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{pmatrix}.$$

Suppose that the reduced row echelon form has a pivot in k columns. What is the dimension of V? Explain how you know this.

$\dim(V) = k$. The colums which have pivots will form a basis for V, because in the vector equation $x_1 \vec{v}_1 + \ldots + x_n \vec{v}_n = \vec{0}$, x_i is free $\iff$ there isn't a pivot on row i.

P4. Let $A = (\vec{v}_1 \cdots \vec{v}_m)$ where $\vec{v}_i$ are vectors in $\mathbb{R}^n$. Suppose that the reduced row echelon form of A has a pivot in every row. Show that we have $\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m) = \mathbb{R}^n$.

Let $\vec{b} \in \mathbb{R}^n$. Then $\vec{b} \in \text{span}(\vec{v}_1, \ldots, \vec{v}_m)$

$\iff \exists \vec{x} \in \mathbb{R}^m \left(A\vec{x} = \vec{b} \right)$.

Since the RREF form has a pivot in every row, $A\vec{x} = \vec{b}$ is consistent.

P5. Determine which of the following sets are bases for $\mathbb{R}^4$.

a) $\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ -1 & 2 & 1 & 1 \end{pmatrix} \xrightarrow[R_4+R_1]{R_2-R_1} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -2 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 4 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow[R_4+2R_2]{R_3-\frac{1}{2}R_2} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & 4 \end{pmatrix} \xrightarrow{R_4-2R_3} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

Basis

b) $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_4+R_1]{R_2-R_1} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3+\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_4-2R_3} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Not a basis

c) $\mathcal{C} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where the $\vec{v}_i$ are any vectors in $\mathbb{R}^4$

$\dim(\mathbb{R}^4) = 4 > 3 \implies$ Not a basis

d) $\mathcal{D} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$, where the $\vec{v}_i$ are any vectors in $\mathbb{R}^4$.

$\dim(\mathbb{R}^4) = 4 < 5 \implies$ Not a basis.