WORKSHEET 4 NAME:

This Worksheet will collected at the end of your recitation section and will be graded on completeness. We will return your graded worksheet back to you during recitation next week.

Chapter 4. Linear Transformations

1. Show that the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ which every vector in $\mathbb{R}^2$ counterclockwise by angle θ is a linear transformation, and find its defining matrix.

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Write $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$, where $r \geq 0$, $0 \leq \phi < 2\pi$. Then

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{pmatrix} = \begin{pmatrix} r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ r(\cos\theta\sin\phi + \cos\phi\sin\theta) \end{pmatrix}$$

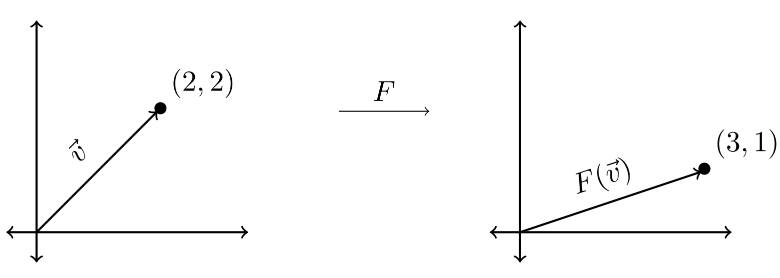
$$= \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{A} \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

This shows that F is linear w/ matrix A

2. Recall that linear transformations map vectors to vectors, but we can also view these transformations as mapping points to points. For example, if F is a linear transformation with

$$F\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

we can also think of F as being a function that moves the point $(2, 2)$ to the point $(3, 1)$.



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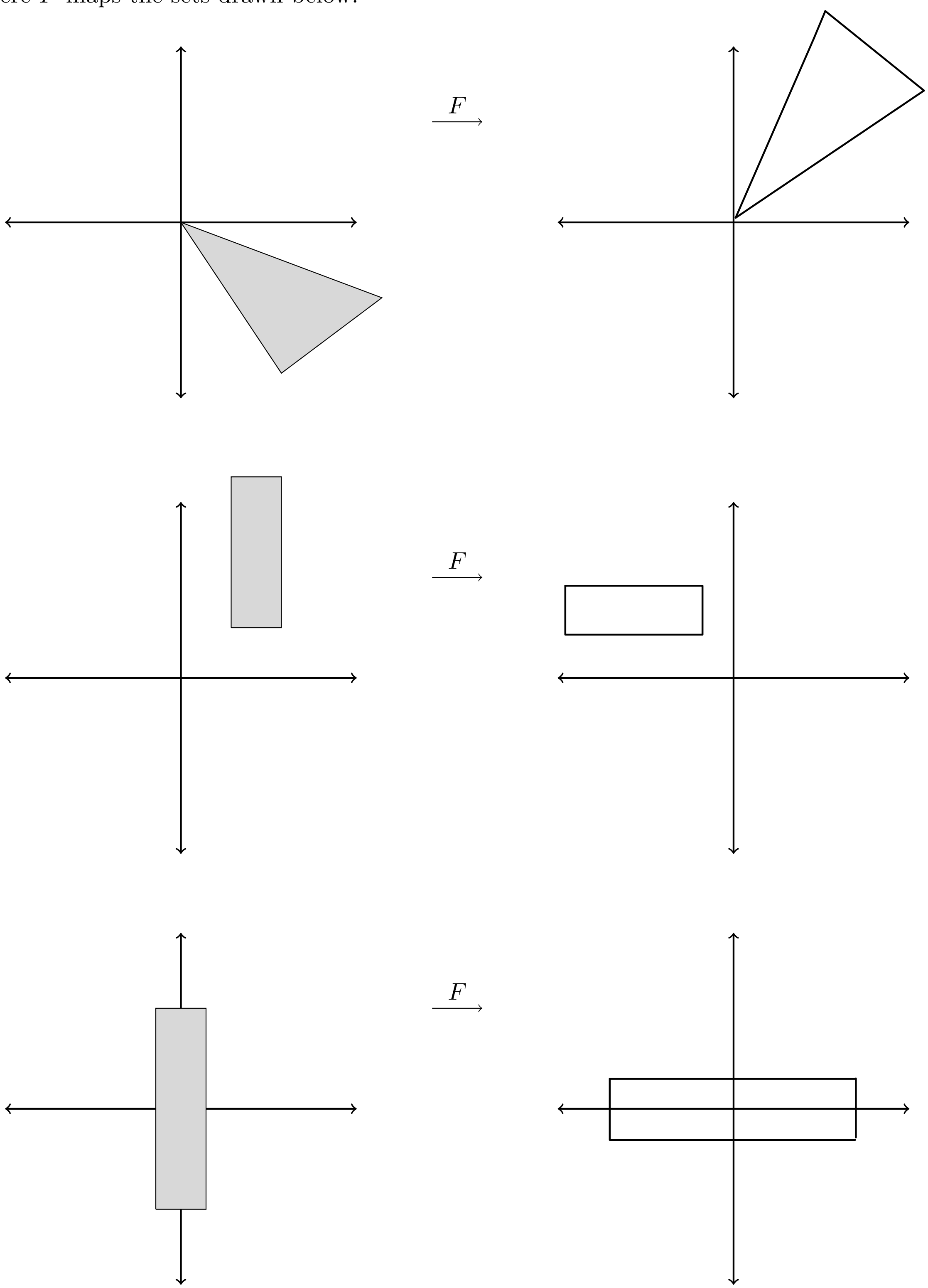
With this perspective, we can also think of linear transformations as mapping sets in $\mathbb{R}^2$ (which are collections of points) to sets in $\mathbb{R}^2$. Let F be the linear transformation which rotates every vector in $\mathbb{R}^2$ counterclockwise by an angle of $90°$. Sketch a rough picture of where F maps the sets drawn below.

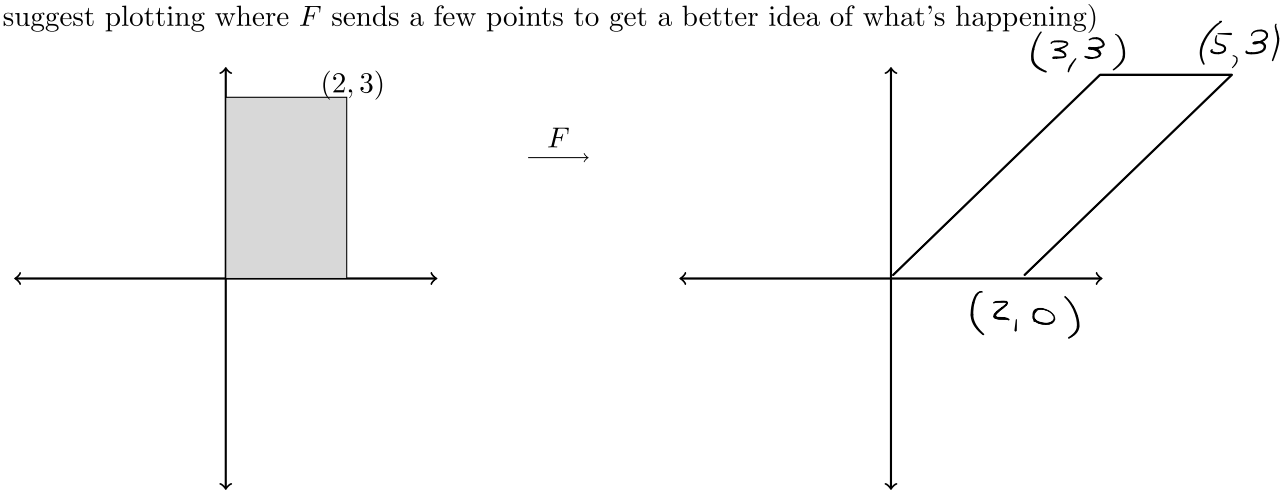


3. A *horizontal shear* is a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ so that

$$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + my \\ y \end{pmatrix}$$

for a constant m called the *shear factor* of F.

a) Sketch a rough picture of where F maps the set drawn below, assuming that $m = 1$. (I'd suggest plotting where F sends a few points to get a better idea of what's happening)



b) Show that every horizontal shear function is a linear transformation, and find its defining matrix (this should depend on m).

Let $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$

$$\Rightarrow F\left(\begin{pmatrix} x \\ y \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \end{pmatrix}\right) = F\begin{pmatrix} x + \lambda a \\ y + \lambda b \end{pmatrix}$$

$$= \begin{pmatrix} x + \lambda a + m(y + \lambda b) \\ y + \lambda b \end{pmatrix} = \begin{pmatrix} x + my \\ y \end{pmatrix} + \lambda \begin{pmatrix} a + mb \\ b \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} + F\begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow F$ is linear.

The matrix of F is:

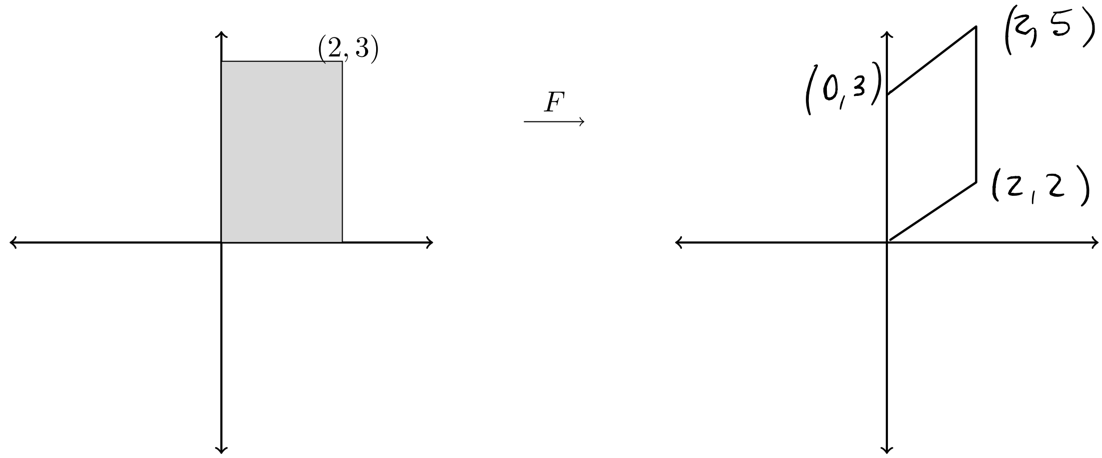
$$\left(F\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad F\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

4. A *vertical shear* is a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ so that

$$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y + mx \end{pmatrix}$$

for a constant m called the *shear factor* of F.

a) Sketch a rough picture of where F maps the set drawn below, assuming that $m = 1$. (I'd suggest plotting where F sends a few points to get a better idea of what's happening)



b) Show that every vertical shear function is a linear transformation, and find its defining matrix (this should depend on m).

Let $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, then

$$F\left(\begin{pmatrix} x \\ y \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \end{pmatrix}\right) = F\begin{pmatrix} x + \lambda a \\ y + \lambda b \end{pmatrix} = \begin{pmatrix} x + \lambda a \\ y + \lambda b + m(x + \lambda a) \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y + mx \end{pmatrix} + \lambda \begin{pmatrix} a \\ a + mb \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} + \lambda F\begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow F$ is linear. The matrix of F is

$$\left(F\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; F\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$$

$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

5. Here are two identities that will be useful to clean up your answers in the following problems:

$$\cos(\theta + \pi/2) = -\sin(\theta), \ \sin(\theta + \pi/2) = \cos(\theta).$$

a) Find a function $F : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates the xy-plane counterclockwise by an angle of θ and leaves the z-axis fixed. You may assume that F is known to be linear.

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xF\vec{e}_1 + yF\vec{e}_2 + zF\vec{e}_3 = x\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + y\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{pmatrix}$$

For $F\vec{e}_2$, note that $F\vec{e}_2 = F\begin{pmatrix} \cos\frac{\pi}{2} \\ \sin\frac{\pi}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{2} + \theta) \\ \sin(\frac{\pi}{2} + \theta) \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$

b) Find a function $G : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates the xz-plane counterclockwise by an angle of θ and leaves the y-axis fixed. You may assume that G is known to be linear.

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xF\vec{e}_1 + yF\vec{e}_2 + zF\vec{e}_3 = x\begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} x\sin\theta + z\cos\theta \\ y \\ x\cos\theta - z\sin\theta \end{pmatrix}$$

This is justified as in (a).

c) Find a function $H : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates the yz-plane counterclockwise by an angle of θ and leaves the x-axis fixed. You may assume that H is known to be linear.

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xF\vec{e}_1 + yF\vec{e}_2 + zF\vec{e}_3 = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} + z\begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y\cos\theta - z\sin\theta \\ y\sin\theta + z\cos\theta \end{pmatrix}$$

Questionnaire:

Below are a few questions which are completely optional, and are meant to benefit you. Please only fill out what you feel comfortable with.

1. What did you think of the worksheet this week (length, difficulty, etc)?

2. Did you feel you worked well with your group this week?

3. Any other comments?

Grading Rubric:

Participation:	/40
Completeness:	/60