WORKSHEET 5 NAME:

This Worksheet will collected at the end of your recitation section and will be graded on completeness. We will return your graded worksheet back to you during recitation next week.

Chapter 5. Matrix Operations

Definition 1. The TRANSPOSE of an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is the $n \times m$ matrix A^T whose ij entry is equal to a_{ji}. That is,

$$A^\top = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

1. Prove the following properties. *(Hint: For parts (a), (b), and (c), show that the matrices on both sides of each equality have the same ij entry. If you get stuck, try some examples first.)*

   a) For any $m \times n$ matrix A, $(A^\top)^\top = A$.

   b) For any $m \times n$ matrices A and B, $(A + B)^\top = A^\top + B^\top$.

c) For 2×2 matrices A and B, show that $(AB)^\top = B^\top A^\top$.

Optional challenge: show this property holds for any general $m \times n$ matrix A and $n \times \ell$ matrix B. *(Hint: show that the matrices on either side of the equality are of the same size and have the same ij-components).*

d) For any $n \times n$ matrix A and positive integer k, $(A^k)^\top = (A^\top)^k$.

2. Use Problem 1 to show that A is invertible if and only if $A^\top$ is invertible, and

$$(A^\top)^{-1} = (A^{-1})^\top.$$

3. An $n \times n$ matrix A is called SYMMETRIC if $A = A^T$. Use Problems 1 and 2 to prove the following.

    a) If A is symmetric, so is A^{-1}.

    b) If A is symmetric, so is A^k for any positive integer k.

4. The ROW SPACE of an $m \times n$ matrix A is the vector subspace $\mathrm{Row}(A)$ of $\mathbb{R}^n$ defined by $\mathrm{Row}(A) = \mathrm{Col}(A^\top)$. That is, $\mathrm{Row}(A)$ is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.

   a) Find a basis for $\mathrm{Row}(A)$ where

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

   b) Give a justification for the following: if A is row equivalent to B then $\mathrm{Row}(A) = \mathrm{Row}(B)$. An informal justification is completely fine here, just make sure you understand what's going on.

   c) Read the proof of Theorem 5.19 from the course lecture notes, which shows that

$$\dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A)).$$

   There's nothing you need to submit for this part, but if you have questions about the proof please use the time to ask your course TA or chat with your groupmates about it.

Questionnaire:

Below are a few questions which are completely optional, and are meant to benefit you. Please only fill out what you feel comfortable with.

1. What did you think of the worksheet this week (length, difficulty, etc)?

2. Did you feel you worked well with your group this week?

3. Any other comments?

Grading Rubric:

Participation:	/40
Completeness:	/60