This Worksheet will be collected at the end of your recitation section and graded on completeness. We will return your graded worksheet back to you during recitation next week.

Chapter 5. Matrix Operations

Definition 1. The TRANSPOSE of an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is the $n \times m$ matrix A^T whose ij entry is equal to a_{ji}. That is,

$$A^\top = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

1. Prove the following properties.

   a) For any $m \times n$ matrix, $(A^\top)^\top = A$.

   Proof. Let A be the $m \times n$ matrix with ij entry equal to a_{ij}. Then, $A^\top$ is the $n \times m$ matrix with ij entry a_{ji} and so $(A^\top)^\top$ is the $m \times n$ matrix with ij entry a_{ij}. Hence, $(A^\top)^\top = A$. □

   b) For any $m \times n$ matrices A and B, $(A + B)^\top = A^\top + B^\top$.

   Proof. If A has ij-entry a_{ij} and B has ij-entry b_{ij} then $A + B$ has ij-entry $a_{ij} + b_{ij}$. So $(A + B)^\top$ is the $n \times m$ matrix with ij-entry $a_{ji} + b_{ji}$, which is precisely the matrix $A^\top + B^\top$. □

c) For 2×2 matrices A and B, show that $(AB)^\top = B^\top A^\top$. Optional challenge: show this property holds for any general $m \times n$ matrix A and $n \times \ell$ matrix B. *(Hint for the general case: show that the matrices on either side of the equality are of the same size and have the same ij-components).*

Solution. I'll just go over the general case here. Please come see one of us in office hour if you'd like help with the 2×2 case.

Proof. Suppose that A is the $m \times n$ matrix with ij entry a_{ij} and B is the $n \times \ell$ matrix with ij entry b_{ij}. Then, by definition of the matrix product, AB is an $m \times \ell$ matrix with ij entry equal to c_{ij} where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{nj}.$$

So by definition of the matrix transpose, $(AB)^\top$ is an $\ell \times m$ matrix with ij-entry

$$c_{ji} = a_{1i}b_{j1} + a_{2i}b_{j2} + \cdots + a_{mi}b_{jn}.$$

Since $B^\top$ is an $\ell \times n$ matrix with ij entry b_{ji} and $A^\top$ is an $n \times m$ matrix with ij entry a_{ji}, then $B^\top A^\top$ is an $\ell \times m$ matrix with ij entry

$$b_{j1}a_{1i} + b_{j2}a_{2i} + \cdots + b_{jn}a_{mi}$$

which is equal to c_{ji}, and from above this is the ij entry of $(AB)^\top$. Hence, $(AB)^\top = B^\top A^\top$. □

d) For any $n \times n$ matrix A and positive integer k, $(A^k)^\top = (A^\top)^k$.

Proof. By part (b), we know that

$$(A^k)^\top = (\underbrace{A \cdot A \cdots A}_{k \text{ times}})^\top = \underbrace{A^\top \cdot A^\top \cdots A^\top}_{k \text{ times}} = (A^\top)^k.$$

□

2. Use Problem 1 to show that A is invertible if and only if $A^\top$ is invertible and

$$(A^\top)^{-1} = (A^{-1})^\top.$$

Proof. Suppose that A is invertible. By Problem 1(c) we get

$$A^\top (A^{-1})^\top = (A^{-1}A)^\top = I_n^\top = I_n,$$

and similarly

$$(A^{-1})^\top A^\top = (AA^{-1})^\top = I_n^\top = I_n.$$

So, $A^\top$ is invertible with inverse $(A^{-1})^\top$.

Conversely, if $A^\top$ is invertible, then by the previous direction $(A^\top)^\top$ is also invertible. But by Problem 1(a) we have $A = (A^\top)^\top$, so A is invertible. □

3. An $n \times n$ matrix A is called SYMMETRIC if $A = A^T$. Use Problems 1 and 2 to prove the following.

   a) If A is symmetric, so is A^{-1}.

   Proof. Since A is symmetric, we have $A^\top = A$. So, by Problem 2 we get

   $$(A^{-1})^\top = (A^\top)^{-1} = A^{-1},$$

   and so A^{-1} is also symmetric. □

   b) If A is symmetric, so is A^k for any positive integer k.

   Proof. Since A is symmetric, we have $A^\top = A$. So, by Problem 1(c) we get

   $$(A^k)^\top = (A^\top)^k = A^k,$$

   and so A^k is symmetric. □

4. The ROW SPACE of an $m \times n$ matrix A is the vector subspace $\mathrm{Row}(A)$ of $\mathbb{R}^n$ defined by $\mathrm{Row}(A) = \mathrm{Col}(A^\top)$. That is, $\mathrm{Row}(A)$ is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.

   a) Find a basis for $\mathrm{Row}(A)$ where

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Proof. Upon row reducing A, we obtain the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

By inspection, the rows of B are linearly independent, therefore the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 3 \end{pmatrix} \right\}$$

is a basis for the row space of B, hence for the row space of A because $\mathrm{Row}(A) = \mathrm{Row}(B)$ (see part (b)). $\square$

   b) Give a justification for the following: if A is row equivalent to B then $\mathrm{Row}(A) = \mathrm{Row}(B)$. An informal justification is completely fine here, just make sure you understand what's going on.

Proof. Let $\vec{a}_1, \dots, \vec{a}_m$ be the rows of A and $\vec{b}_1, \dots, \vec{b}_m$ be the rows of B, so $\mathrm{Row}(A) = \mathrm{span}\{\vec{a}_1, \dots, \vec{a}_m\}$ and $\mathrm{Row}(B) = \mathrm{span}\{\vec{b}_1, \dots, \vec{b}_m\}$.
Each row of B is obtained as a linear combination of rows of A, hence $\vec{b}_i \in \mathrm{Row}(A)$ for every i, and so $\mathrm{Row}(B) \subset \mathrm{Row}(A)$. Moreover, the process can be reversed, so each row of A is a linear combination of rows of B, hence $\vec{a}_i \in \mathrm{Row}(B)$ for every i, and so $\mathrm{Row}(A) \subset \mathrm{Row}(B)$. $\square$

   c) Read the proof of Theorem 5.19 from the course lecture notes, which shows that

$$\dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A)).$$

There's nothing you need to submit for this part, but if you have questions about the proof please use the time to ask your course TA or chat with your groupmates about it.

Questionnaire:

Below are a few questions which are completely optional, and are meant to benefit you. Please only fill out what you feel comfortable with.

1. What did you think of the worksheet this week (length, difficulty, etc)?

2. Did you feel you worked well with your group this week?

3. Any other comments?

Grading Rubric:

Attendance:	/20
Participation:	/20
Completeness:	/60