HOMEWORK 8 21-441, FALL 2022

Instructions: Complete all problems from the list below. This assignment will be due on Gradescope no later than **7pm on Wednesday, November 16th**. Late work will not be accepted. There will be no exceptions for technology issues, so I suggest you upload your homework at least one hour before the deadline. Please make sure you've done all of the following before submitting your work:

* **Do not** write your name anywhere on your submission. Gradescope will keep track of your submission, and will allow me to use a blind grading process.
* Type your homework using LaTeX.
* Write up proofs formally and completely.
* If you use any resources (stackexchange, tutors, friends), please include a list of references in your writeup.

Chapter 5 Problems:

6. Let K be a number field with $[K : \mathbb{Q}] = n$. Prove the following.

   a) For any $p, q \in \mathbb{Q}$ and $\alpha, \beta \in K$

   $$N_K(p\,\alpha\beta) = p^n N_K(\alpha) N_K(\beta), \text{ and}$$
   $$\mathrm{Tr}_K(p\alpha + q\beta) = p\mathrm{Tr}_K(\alpha) + q\mathrm{Tr}_K(\beta).$$

   b) If $\alpha \in K$ is of degree m, then

   $$N_K(\alpha) = (N_{\mathbb{Q}(\alpha)}(\alpha))^d, \text{ and}$$
   $$\mathrm{Tr}_K(\alpha) = d\mathrm{Tr}_{\mathbb{Q}(\alpha)}(\alpha),$$

   where $d = n/m$.

   c) An element $u \in \mathcal{O}_K$ is a unit (that is, u has a multiplicative inverse) if and only if $N_K(u) = \pm 1$.

   d) Show that for $\pi \in \mathcal{O}_K$, if $N_K(\pi)$ is a rational prime then π is irreducible in $\mathcal{O}_K$.

7. Show that $\Delta(\alpha_1, \ldots, \alpha_n) = 0$ if and only if the α_i are linearly dependent.

8. Find the ring of integers for **any two** of the following number fields.

   a) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

   b) $\mathbb{Q}(\sqrt{2}, i)$

   c) $\mathbb{Q}(\sqrt[3]{2})$

   d) $\mathbb{Q}(\sqrt[4]{2})$