SAMS 2023 WEEK 1: DIVISIBILITY

Please keep this packet and bring it with you to class every day this week to work on. This packet will not be collected, but I encourage you to use our google group (linked on our course webpage) to post questions and solutions. A list of mini projects will be included in next week's worksheet, and the first round of presentations will take place next Friday (July 14) at the beginning of class.

Definitions and Theorems

The following definitions and theorems will be introduced during lecture, and will be needed for this week's problem set. Note that a *definition* is some explanation of the meaning of a word. A *theorem* is some statement which has been demonstrated to be true.

Definition 1. Suppose that a and b are integers. We say that a DIVIDES b if there exists an integer k so that $b = ak$. In this case, we write $a \mid b$. If a does not divide b, we write $a \nmid b$.

Definition 2. An integer a is EVEN if it can be written in the form $a = 2k$ for some integer k.

Definition 3. An integer b is ODD if it can be written in the form $b = 2\ell + 1$ for some integer ℓ.

Theorem 1 (The Division Algorithm)**.** Suppose that a and b are integers, and that $a > 0$. Then, there exist *unique* integers q and r so that

$$b = qa + r$$

where $0 \leq r < a$. We call q the *quotient* and r the *remainder* when dividing b by a.

Problem Set

Complete as many problems from the list below as you have time and interest for. Feel free to skip around as you'd like, and to work on your own or with your group as you prefer. If you generally prefer to work on your own, I encourage you to discuss at least two problems together with your group. I suggest you keep a notebook or binder for this course to store your solutions. There is also scratch paper available at the front of the class for you to use at any time.

P1. Use the formal definition of divisibility (Definition 1) to show the following

$$3 \mid 99, \;\; 13 \mid 1001, \;\; -5 \mid 500, \;\; \text{and} \;\; 22 \mid -1716.$$

P2. Find all integers (positive **and negative**) that divide 66.

P3. Explain why we know that $10 \nmid 5$.

P4. Suppose that for integers a and b we have $a > b$. Is it possible for $a \mid b$? Why or why not?

P5. Let's explore some properties of integers divisible by 6.

   a) List five examples of integers divisible by 6. Use the notation $6 \mid a$ given in Definition 1 in each of your examples.

b) Justify each of your examples above using the formal definition of integer divisibility (Definition 1).

c) Look at your list of examples from above. Is there another integer which divides each of your examples besides 6?

d) Formulate a *conjecture* (that is, something you believe to be true) of the form: " if a is an integer divisible by 6, then a is divisible by (blank)".

e) Convince the members of your group that your conjecture from the previous part is true. Try using the formal definition of divisibility in your justification.

P6. Next, let a and b be any integers with $a \mid b$. Do you think the following statement is true: if d divides a then d divides b? Why or why not? Can you convince your group members of your answer?

P7. What integers divide 1? What integers divide 0? Justify your answers.

P8. Suppose that a and b are integers with $a \mid b$ and $b \mid a$. Is it always going to be the case that $a = b$? Why or why not?

P9. Let a, b, and c be integers. Use the formal definition of divisibility to justify each of the following properties.

a) If $a \mid b$ and $a \mid c$ then $a \mid (b + c)$

b) If $a \mid b$ and $a \mid c$ then $a \mid (b - c)$

P10. For the following pairs of integers, use the division algorithm to find the unique quotient and remainder when dividing b by a.

a) $b = 277$, $a = 4$

b) $b = 33$, $a = 11$

c) $b = -48$, $a = 13$

P11. In lecture, we saw that if an integer b is **not** even, then we can write it in the form $b = 2k+1$ for some integer k, which justified our formal definition of odd integers (Definition 3).

a) Choose five odd integers, and write them in the form $2k + 1$ for an integer k.

b) Write the integers from the previous part in the form $2\ell - 1$ for some integer ℓ.

c) Do you think that all odd integers can be written in the form $2\ell - 1$ for some integer ℓ? Why or why not?

P12. Use the formal definition of even and odd integers to explain why each of the following statements are always true.

a) The sum of any two even numbers is even.

b) The sum of any two odd numbers is odd.

c) The sum of an even and an odd number is odd.

P13. Determine which of the following statements are true or false. A statement is true if it holds for **any choice** of integers. A statement is false if there is **at least one choice** of integers where the statement doesn't hold true.

a) True or False? For all integers a, b and c, if $a \mid (b + c)$ then $a \mid b$ or $a \mid c$.

b) True or False? For integers a, b and c, if $a \mid bc$ then $a \mid b$ or $a \mid c$.

c) True or False? The sum of the squares of any two odd integers is even.

Additional Problems

The problem set below requires methods and background we won't necessarily cover in our course. If you've seen proof methods before, or want some extra challenge, feel free to play with these!

A1. Show that the product of any three consecutive integers is divisible by 6.

A2. How many integers between 100 and 1000 are divisible by 7?

A3. How many integers between 100 and 1000 are divisible by 49?

A4. Find the number of positive integers less than 1000 that are not divisible by 2 or 5.

A5. Find the number of positive integers less than 1000 that are not divisible by 2, 5 or 7.

A6. The Fibonacci sequence is the integer linear recurrence sequence given by

$$f_0 = 0, f_1 = 1$$

$$f_{n+2} = f_{n+1} + f_n.$$

Some terms of the Fibonacci sequence are listed below

n	0	1	2	3	4	5	6
f_n	0	1	1	2	3	5	8

a) Show that f_n is even if and only if n is divisible by 3.

b) Show that f_n is divisible by 3 if and only if n is divisible by 4.

c) Show that f_n is divisible by 6 if and only if n is divisible by 6.