SAMS 2023 WEEK 2: DIVISIBILITY CONTINUED

Please keep this packet and bring it with you to class every day this week to work on. This packet will not be collected, but I encourage you to use our google group (linked on our course webpage) to post questions and solutions. A list of mini projects is included at the end of this worksheet. I'll ask groups to select a project topic no later than Wednesday, and to give a short (5-10 minute presentation) at the beginning of class this Friday.

Definitions and Theorems

The following definitions and theorems will be introduced during lecture, and will be needed for this week's problem set. Note that a *definition* is some explanation of the meaning of a word. A *theorem* is some statement which has been demonstrated to be true.

Definition 1. Let a and b be integers. We say that an integer d is a COMMON DIVISOR of a and b if $d \mid a$ and $d \mid b$.

Definition 2. The GREATEST COMMON DIVISOR of integers a and b is the largest integer d so that d is a common divisor of a and b. In this case, we use the notation $d = \gcd(a, b)$.

Definition 3. If $\gcd(a, b) = 1$ we say that a and b are RELATIVELY PRIME.

Theorem 1. Let a and b be integers with $a > 0$. Using the Division Algorithm, write

$$b = aq + r$$

for $0 \leq r < a$. Then $\gcd(b, a) = \gcd(a, r)$.

Problem Set

Complete as many problems from the list below as you have time and interest for. Feel free to skip around as you'd like, and to work on your own or with your group as you prefer. If you generally prefer to work on your own, I encourage you to discuss at least two problems together with your group. I suggest you keep a notebook or binder for this course to store your solutions. There is also scratch paper available at the front of the class for you to use at any time.

P1. Find the following greatest common divisors. Which pairs are relatively prime?

a) $\gcd(18, 12)$
b) $\gcd(66, 22)$
c) $\gcd(0, 21)$
d) $\gcd(0, a)$ for any positive integer a
e) $\gcd(0, -a)$ for any positive integer a
f) $\gcd(0, 0)$
g) $\gcd(1, 14598)$
h) $\gcd(-1, 14598)$

i) $\gcd(1, a)$ for any integer a

j) $\gcd(-1, a)$ for any integer a

P2. Suppose that a and d are positive integers with $d \mid a$. What is $\gcd(a, d)$?

P3. Redo the example from lecture where we showed $\gcd(128, 34) = 2$ with the Euclidean Algorithm. Talk to your group about this process. Do you believe the algorithm will always work? How do we know it must always terminate?

P4. Use the Euclidean Algorithm to find the following greatest common divisors.

a) $\gcd(112, 92)$

b) $\gcd(31, 162)$

c) $\gcd(12, 256)$

d) $\gcd(243, 9)$

P5. Redo the example from lecture where we reversed the Euclidean algorithm to find an integer solution to the linear Diophantine equation

$$128x + 34y = 2.$$

Make sure you understand every step before moving on!

P6. Use your work in P4 to find any one integer solution to the following linear Diophantine equations. If no solutions exist, state why not.

a) $112x + 92y = 4$

b) $112x + 92y = -12$

c) $112x + 92y = 3$

d) $31x + 162y = 1$

e) $31x + 162y = -27$

P7. Suppose that a and b are relatively prime. Show that every linear Diophantine equation

$$ax + by = c$$

has a solution.

P8. Suppose that $\gcd(a, b) > 1$. For what values of c does the linear Diophantine equation

$$ax + by = c$$

have a solution? How do you know this?

P9 For integers a and b, what do you think would be a good definition for the LEAST COMMON MULTIPLE of a and b? Construct and compute a few examples.

Additional Problems

The problem set below requires methods and background we won't necessarily cover in our course. If you've seen proof methods before, or want some extra challenge, feel free to play with these!

A1. Let a, b and c be integers. If $a \mid bc$ and $\gcd(a, b) = 1$ then $a \mid c$.

A2. Let a, b and n be integers. If $a \mid n$, $b \mid n$ and $\gcd(a, b) = 1$ then $ab \mid n$.

A3. Let a, b, and n be integers. If $\gcd(a, n) = 1$ and $\gcd(b, n) = 1$ then $\gcd(ab, n) = 1$.

A4. If a and b are positive integers, show that

$$\gcd(a, b) \cdot lcm(a, b) = ab.$$

A5. If a and b are positive integers, show that

$$lcm(a, b) = ab$$

if and only if a and b are relatively prime.

Mini Projects

Work with your group to select a topic from the list below that looks interesting to investigate. Only one topic may be covered by each group, and sign ups will be on a first come first serve basis. My only instruction for your presentations is to tell us something interesting about what you investigated in a way you're proud of. Your presentations should last about 5 minutes, but please make sure to take **no longer than 10 minutes**. There are no requirements, grades, or judgment for this assignment. Engage with it as much as is interesting to you!

TOPICS.

1. **Perfect numbers**. Some things you might investigate: What are they? What's their history? Can you give some examples of perfect numbers? What's known about them? What do we still not know about them?

2. **Euclid**. Some things you might investigate: Who was Euclid? How do you say his name? What is he best known for? What are *The Elements* and why was it so important?

3. **Diophantus of Alexandria**. Some things you might investigate: Who was Diophantus? How do you say his name? What is he best known for? What is *The Arithmetica* and why was it so important?

4. **Sophie Germain**. Some things you might investigate: Who was she? What were some of her major contributions to number theory? What was it like for her to be a woman mathematician in the 18th and 19th centuries?

5. **Fermat's Last Theorem**. Some things you might investigate: What is this theorem statement? Who was Fermat? Where was this theorem stated, and why did we believe it was true for so long without proof? How long did it take to prove?

6. **The Babylonian Number System.** Some things you might investigate: What was this? What's the history? Tell us how this gave us the number of seconds in a minute or minutes in an hour.