SAMS 2023 WEEK 3: PRIME NUMBERS

Please keep this packet and bring it with you to class every day this week to work on. This packet will not be collected, but I encourage you to use our google group (linked on our course webpage) to post questions and solutions. A list of mini projects is included at the end of this worksheet. I'll ask groups to select a project topic no later than Wednesday, and to give a short (5-10 minute presentation) at the beginning of class this Friday.

Definitions and Theorems

The following definitions and theorems will be introduced during lecture, and will be needed for this week's problem set. Note that a *definition* is some explanation of the meaning of a word. A *theorem* is some statement which has been demonstrated to be true.

Definition 1. An integer $p > 1$ is PRIME if the only positive divisors of p are 1 and p.

Theorem 1 (The Fundamental Theorem of Arithmetic)**.** Let n be a positive integer. Then, there exist unique prime numbers $p_1, p_2, \ldots, p_t$ and integers $k_1, \ldots, k_t > 0$ so that

$$n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$$

where the primes p_i are distinct and ordered so that $p_1 < p_2 < \cdots < p_t$.

Theorem 2. Suppose that a positive integer n has prime factorization $n = p_1^{k_1} \cdots p_t^{k_t}$. Then, the divisors of n are given by $n = p_1^{\ell_1} \cdots p_t^{\ell_t}$ where $0 \leq \ell_i \leq k_i$ for all $i = 1, \ldots, t$.

Problem Set

Complete as many problems from the list below as you have time and interest for. Feel free to skip around as you'd like, and to work on your own or with your group as you prefer. If you generally prefer to work on your own, I encourage you to discuss at least two problems together with your group. I suggest you keep a notebook or binder for this course to store your solutions. There is also scratch paper available at the front of the class for you to use at any time.

P1. Find all of the prime numbers between 1 and 30.

P2. Show that 2 is the only even prime number.

P3. Find the unique prime factorization of the following integers.

- a) 126
- b) 1617
- c) 4554

P4. Use the Fundamental Theorem of Arithmetic to find all divisors of the integers from P3.

P5. Use the Fundamental Theorem of Arithmetic to find the greatest common divisors of the following pairs of integers.

a) $\gcd(126, 1617)$
b) $\gcd(1617, 4554)$
c) $\gcd(126, 4554)$

P6. Suppose that a and b are integers with prime factorizations

$$a = p_1^2\, p_2^5\, p_3$$

$$b = p_1\, p_2^2\, p_3^4 p_4.$$

What is $\gcd(a, b)$?

P7. Without using a calculator, determine the number of zeros at the end of 25!, where

$$25! = 25 \cdot 24 \cdots 3 \cdot 2 \cdot 1$$

is the FACTORIAL of 25. *(Hint: use the Fundamental Theorem of Arithmetic).*

Additional Problems

The problem set below requires methods and background we won't necessarily cover in our course. If you've seen proof methods before, or want some extra challenge, feel free to play with these!

A1. Show that there do not exist natural numbers m, n so that $7m^2 = n^2$.

A2. Show that there do not exist natural numbers m, n so that $24m^3 = n^3$

A3. Let p be prime and a, b be integers. Show that if $p \mid ab$ then $p \mid a$ or $p \mid b$.

A4. Let n be any integer. Is it true that if $n \mid ab$ then $n \mid a$ or $n \mid b$?

A5. Recall that a RATIONAL NUMBER is any number of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. Use the Fundamental Theorem of Arithmetic to show that $\sqrt{2}$ is not rational.

Mini Projects

Work with your group to select a topic from the list below that looks interesting to investigate. Only one topic may be covered by each group, and sign ups will be on a first come first serve basis. My only instruction for your presentations is to tell us something interesting about what you investigated in a way you're proud of. Your presentations should last about 5 minutes, but please make sure to take **no longer than 10 minutes**. There are no requirements, grades, or judgment for this assignment. Engage with it as much as is interesting to you!

TOPICS.

1. **Mersenne Primes**. Some things you might investigate: Recall their connection to perfect numbers. Lookup the Great Internet Mersenne Prime Search (GIMP), there's a cash prize in this story somewhere. Tell us the largest known prime number. Who found it?

2. **Sieve of Eratosthenes**. Some things you might investigate: Who was Eratosthenes? Explain how the sieve of Eratosthenes works. Use this method to find all prime numbers between 1 and 100 by hand.

3. **Twin Primes**. Some things you might investigate: What are they? Can you list a few examples of twin primes? What is the Twin Primes Conjecture? What is the current state of the Twin Primes Conjecture? What mathematicians have made progress on studying this conjecture?

4. **The Goldbach Conjecture**. Some things you might investigate: What does this conjecture say? Who was Goldbach? What is the current state of the Goldbach conjecture?

5. **Wilson's Theorem**. Some things you might investigate: What does this theorem state? Who was Ibn al-Haytham? Who was John Wilson? Who else was involved in discovering or proving this theorem? Use Wilson's Theorem to check that a large prime number is in fact prime.

6. **Fermat Primes**. Some things you might investigate: What is a Fermat prime? List the first five Fermat numbers. State Fermat's conjecture that all Fermat numbers are prime. Was he correct? Investigate what we know about Fermat primes.