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A Geometric Perspective on Lifting

Michele Conforti
Dipartimento di Matematica Pura ed Applicata, Università di Padova, 35121 Padova, Italy, conforti@math.unipd.it
Gérard Cornuéjols
Tepper School of Business, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213, gc0v@andrew.cmu.edu
Giacomo Zambelli
London School of Economics and Political Sciences, London WC2A 2AE, United Kingdom, g.zambelli@lse.ac.uk

Recently it has been shown that minimal inequalities for a continuous relaxation of mixed-integer linear programs are associated with maximal lattice-free convex sets. In this paper, we show how to lift these inequalities for integral nonbasic variables by considering maximal lattice-free convex sets in a higher dimensional space. We apply this approach to several examples. In particular, we identify cases in which the lifting is unique.

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1. Introduction

A classical topic in integer programming is that of *lifting*, introduced by Padberg (1973): given mixed-integer sets $Q \subset \mathbb{R}^n$ and $R \subset \mathbb{R}^{n+p}$ such that Q is the restriction of R obtained by setting the last p variables to 0, and given a valid inequality $\sum_{i=1}^{n} a_j x_j \leqslant b$ for Q, find coefficients $a_{n+1}, \ldots, a_{n+p}$ such that $\sum_{i=1}^{n+p} a_j x_j \leqslant b$ is valid for R. Current state-of-the-art integer programming solvers routinely use lifted knapsack covers, lifted flow covers, and other liftings. The lifting coefficients $a_{n+1}, \ldots, a_{n+p}$ can be computed sequentially, choosing the best possible value at each step. However, different orderings of the variables usually lead to different answers. An aspect of liftings that has received attention is that of sequence-independent lifting (Wolsey 1977, Gu et al. 2000, Atamtürk 2004). In this paper, we revisit liftings from a geometric perspective, building on recent work relating minimal inequalities to maximal lattice-free convex sets. Our results are best described in the context of an infinite model, which we present next.

Let S be the set of integral points in some rational polyhedron in $\mathbb{R}^n$ such that $\dim(\text{conv}(S)) = n$ (for example S could be the set of nonnegative integral points), and let $f \in \text{conv}(S)\backslash\mathbb{Z}^n$. We consider the following semi-infinite model:

$$\begin{aligned} &x = f + \sum_{r\in\mathbb{R}^n} r s_r + \sum_{r\in\mathbb{R}^n} r y_r,\\ &x \in S,\\ &s_r \geqslant 0, \quad r \in \mathbb{R}^n,\\ &y_r \geqslant 0, \quad y_r \in \mathbb{Z}, \quad r \in \mathbb{R}^n,\\ &s, y \text{ have finite support.}\end{aligned} \tag{1}$$

The infinite vectors s and y having *finite support* means that they are nonzero only in a finite number of entries. Given two functions ψ and π from $\mathbb{R}^n$ to $\mathbb{R}$, the inequality

$$\sum_{r\in\mathbb{R}^n} \psi(r) s_r + \sum_{r\in\mathbb{R}^n} \pi(r) y_r \geqslant 1 \tag{2}$$

is *valid* for (1) if it holds for every (x, s, y) satisfying (1). If (2) is valid, we say that the function (ψ, π) is *valid* for (1). A valid function (ψ, π) is *minimal* if there is no valid function (ψ', π') distinct from (ψ, π) such that $\psi'(r) \leqslant \psi(r)$, $\pi'(r) \leqslant \pi(r)$ for all $r \in \mathbb{R}^n$.

Model (1) is a natural abstraction of the simplex tableau. Indeed, setting all but a finite number of the s_r and y_r variables to zero reduces (1) to a problem in tableau form with right-hand-side f, where x are the basic variables, and the s_r and y_r variables not set to zero are the nonbasic ones. Therefore, information about valid inequalities for (1) automatically transfers to the problem of cutting off a fractional basic solution of the linear programming relaxation. Most cutting planes used in practice (Gomory mixed-integer cuts, mixed-integer rounding inequalities, knapsack covers, flow covers, lift-and-project cuts, and many others) are valid for Gomory's corner polyhedron, which is the convex hull of solutions to (1) where $S = \mathbb{Z}^n$ and all but a finite number of the variables s_r and y_r are set to 0.

One of the most effective cutting planes used in solvers are the Gomory mixed-integer cuts, which correspond to valid functions for (1) when $n = 1$ and $S = \mathbb{Z}$. It is well known that, among all cutting planes derived from a single equation, Gomory mixed-integer cuts have the best possible coefficients (i.e., the smallest) on the nonbasic continuous variables. To transfer this notion to the general setting of (1), Dey and Wolsey (2010) proposed to study the following simpler model, where the integer variables y_r are all set to zero.

$$\begin{aligned} & x = f + \sum_{r\in\mathbb{R}^n} r s_r, \\ & x \in S, \\ & s_r \geqslant 0, \quad r \in \mathbb{R}^n, \\ & s \text{ has finite support.} \end{aligned} \tag{3}$$

We refer to this model as the *continuous semi-infinite relaxation relative to* f. Given a valid function ψ for (3), the function π is a *lifting* of ψ if (ψ, π) is valid for (1). If ψ is a minimal valid function for (3) and π is a lifting of ψ such that (ψ, π) is minimal, we say that π is a *minimal lifting* of ψ.

Minimal valid inequalities for (3) are well understood in terms of maximal S-free convex sets. We are interested in characterizing liftings of minimal valid inequalities for (3).

We remark that, given any valid function ψ for (3) and a lifting π of ψ, the function π' defined by $\pi'(r) = \min\{\psi(r), \pi(r)\}$ is also a lifting of ψ. Indeed, given $(\bar{s}, \bar{y})$ satisfying (1), we show that

$$\sum_{r\in\mathbb{R}^n} \psi(r)\bar{s}_r + \sum_{r\in\mathbb{R}^n} \pi'(r)\bar{y}_r \geqslant 1.$$

Let $(\tilde{s}, \tilde{y})$ be defined by $\tilde{s}_r = \bar{s}_r$, $\tilde{y}_r = \bar{y}_r$ for every $r \in \mathbb{R}^n$ such that $\pi(r) \leqslant \psi(r)$, and $\tilde{s}_r = \bar{s}_r + \bar{y}_r$, $\tilde{y}_r = 0$ for every $r \in \mathbb{R}^n$ such that $\psi(r) < \pi(r)$. One can readily verify that $(\tilde{s}, \tilde{y})$ satisfies (1), hence $\sum_{r\in\mathbb{R}^n} \psi(r)\tilde{s}_r + \sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r \geqslant 1$. Furthermore,

$$\sum_{r\in\mathbb{R}^n} \psi(r)\bar{s}_r + \sum_{r\in\mathbb{R}^n} \pi'(r)\bar{y}_r = \sum_{r\in\mathbb{R}^n} \psi(r)\tilde{s}_r + \sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r \geqslant 1.$$

In particular, if ψ is a minimal valid function for (3) and π is a minimal lifting of ψ, then $\pi \leqslant \psi$.

We first concentrate on deriving the best possible lifting coefficient of one single integer variable. Namely, given $d \in \mathbb{R}^n$, we consider the model

$$\begin{aligned} & x = f + \sum_{r\in\mathbb{R}^n} r s_r + dz, \\ & x \in S, \\ & s_r \geqslant 0, \quad r \in \mathbb{R}^n, \\ & z \geqslant 0, \quad z \in \mathbb{Z}, \\ & s \text{ has finite support.} \end{aligned} \tag{4}$$

Given a minimal valid function ψ for (3), let $\pi_l(d)$ be the minimum scalar λ such that the inequality

$$\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geqslant 1$$

is valid for (4).

By definition, $\pi_l \leqslant \pi$ for every lifting π of ψ. In general, the function (ψ, π_l) is not valid for (1). However, when (ψ, π_l) is valid, π_l can be viewed as a trivial sequence-independent lifting of ψ.

Proposition 1. *Let ψ be a minimal valid function for* (3). *When (ψ, π_l) is valid for* (1), *π_l is the unique minimal lifting of ψ.*

In this paper we give a geometric characterization of the function π_l, and we use this characterization to analyze specific minimal valid functions ψ for which π_l is the unique minimal lifting.

A valid function (ψ, π) is *extreme* for (1) if there do not exist distinct valid functions (ψ^1, π^1), (ψ^2, π^2) such that $(\psi, \pi) = \frac{1}{2}(\psi^1, \pi^1) + \frac{1}{2}(\psi^2, \pi^2)$. Note that if ψ is extreme for (3), then ψ is minimal.

Remark 2. If ψ is extreme for (3) and (ψ, π_l) is valid for (1), then (ψ, π_l) is extreme for (1).

Indeed, given valid functions (ψ^1, π^1), (ψ^2, π^2) such that $(\psi, \pi) = \frac{1}{2}(\psi^1, \pi^1) + \frac{1}{2}(\psi^2, \pi^2)$, then $\psi_1 = \psi_2 = \psi$, since ψ is extreme for (3), and $\pi_1 = \pi_2 = \pi_l$ since $\pi_1 \geqslant \pi_l$ and $\pi_2 \geqslant \pi_l$.

2. Lifting and S-Free Convex Sets

We observe that (4) is equivalent to the following

$$\begin{aligned} & \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} + \sum_{r\in\mathbb{R}^n} \begin{pmatrix} r \\ 0 \end{pmatrix} s_r + \begin{pmatrix} d \\ 1 \end{pmatrix} z \\ & (x, x_{n+1}) \in S \times \mathbb{Z}_+ \\ & s_r \geqslant 0, \quad r \in \mathbb{R}^n \\ & z \geqslant 0, \\ & s \text{ has finite support.} \end{aligned} \tag{5}$$

Indeed (x, s, z) is a solution for (4) if and only if (x, x_{n+1}, s, z) is a solution to (5) by setting $x_{n+1} = z$. Note that the above is obtained from the continuous semi-infinite relaxation relative to $\binom{f}{0}$ by setting to 0 all variables relative to rays with nonzero $(n+1)$th component, except for $\binom{d}{1}$. Therefore, given any valid function $\bar{\psi}$ for the continuous semi-infinite relaxation relative to $\binom{f}{0}$, then if we let $\psi(r) = \bar{\psi}\binom{r}{0}$ for $r \in \mathbb{R}^n$ and $\lambda = \bar{\psi}\binom{d}{1}$, the inequality $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geqslant 1$ is valid for (5) and for (4).

A convex set is *S-free* if it does not contain any point of S in its interior. Maximal S-free convex sets were characterized in Basu et al. (2010b), where it was also shown that there is a one-to-one correspondence between minimal valid functions for (3) and maximal S-free convex sets with f in their interior.

Theorem 3 (Basu et al. 2010b). *A full-dimensional convex set B is a maximal S-free convex set if and only if it is a polyhedron such that B does not contain any point of S in its interior and each facet of B contains a point of S in its relative interior. Furthermore, if $B \cap \operatorname{conv}(S)$ has nonempty interior,* $\operatorname{lin}(B)$ *contains* $\operatorname{rec}(B \cap \operatorname{conv}(S))$.

We explain how minimal valid inequalities for (3) arise from maximal S-free convex sets. Let B be a polyhedron with f in its interior, and let $a_1, \ldots, a_t \in \mathbb{R}^q$ such that $B = \{x \in \mathbb{R}^n \mid a_i(x-f) \leqslant 1,\ i = 1\ldots, t\}$. We define the function $\psi_B\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\psi_B(r) = \max_{i=1,\ldots,t} a_i r.$$

Note that the function ψ_B is convex, *subadditive*, i.e., $\psi_B(r) + \psi_B(r') \geqslant \psi_B(r+r')$, and *positively homogeneous*, i.e., $\psi_B(\lambda r) = \lambda\psi_B(r)$ for every $\lambda \geqslant 0$.

We claim that if B is a maximal S-free convex set, then

$$\sum_{r\in\mathbb{R}^n} \psi_B(r)s_r \geqslant 1 \quad \text{is valid for (3).} \tag{6}$$

Indeed, let (x, s) be a solution of (3). Note that $x \in S$, thus $x \notin \mathbf{int}(B)$. Then

$$\sum_{r\in\mathbb{R}^n} \psi_B(r)s_r = \sum_{r\in\mathbb{R}^n} \psi_B(rs_r) \geqslant \psi_B\left(\sum_{r\in\mathbb{R}^n} rs_r\right) = \psi_B(x-f) \geqslant 1,$$

where the first equation follows from positive homogeneity, the first inequality follows from subadditivity of ψ_B, and the last one follows from the fact that $x \notin \mathbf{int}(B)$.

The above functions are minimal (Dey and Wolsey 2010, Basu et al. 2010b). It was proved in Basu et al. (2010b) that the converse is also true, namely that every minimal function valid for (3) is of the form ψ_B where B is a maximal S-free convex set with f in its interior.

Example. We consider problem (1) when $n = 1$, $0 < f < 1$, and $S = \mathbb{Z}$. In this case the only maximal S-free convex set containing f is the interval $B = [0, 1]$. Thus $B = \{x \in \mathbb{R} \mid -f^{-1}(x-f) \leqslant 1,\ (1-f)^{-1}(x-f) \leqslant 1\}$, and $\psi_B(r) = \max\{-f^{-1}r, (1-f)^{-1}r\}$.

Let ψ be a minimal valid function for (3), and let $B = \{x \in \mathbb{R}^n \mid a_i(x-f) \leqslant 1,\ i = 1, \ldots, t\}$ be a maximal S-free convex set with f in its interior such that $\psi = \psi_B$. We define the set $B(\lambda) \subset \mathbb{R}^{n+1}$ as follows:

$$B(\lambda) = \big\{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid a_i(x-f) + (\lambda - a_i d)x_{n+1} \leqslant 1,\ i = 1, \ldots, t\big\}. \tag{7}$$

Theorem 4. *The inequality $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geqslant 1$ is valid for (4) if and only if $B(\lambda)$ is $(S \times \mathbb{Z}_+)$-free.*

Proof. Let $\bar\psi = \psi_{B(\lambda)}$. By construction, $\bar\psi\binom{r}{0} = \psi(r)$ for all $r \in \mathbb{R}^n$, while $\bar\psi\binom{d}{1} = \lambda$.

We show the "if" part of the statement. Given λ such that $B(\lambda)$ is $(S \times \mathbb{Z}_+)$-free, it follows by claim (6) that the function $\bar\psi$ is valid for the continuous semi-infinite relaxation relative to $\binom{f}{0}$. This implies that $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geqslant 1$ is valid for (4).

We now prove the "only if" part. Let λ be such that $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geqslant 1$ is valid for (4). Given a point $\binom{\bar x}{\bar x_{n+1}} \in S \times \mathbb{Z}_+$, we show that such point is not in the interior of $B(\lambda)$. Indeed, let $\bar r = \bar x - \bar x_{n+1}d - f$, $\bar z = \bar x_{n+1}$, and $(\bar s_r)_{r\in\mathbb{R}^n}$ be defined by

$$\bar s_r = \begin{cases} 1 & \text{if } r = \bar r, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f + \sum_{r\in\mathbb{R}^n} r\bar s_r + d\bar z = f + \bar r + \bar x_{n+1}d = \bar x$. Because $\bar x \in S$ and $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geqslant 1$ is valid for (4), we have

$$1 \leqslant \sum_{r\in\mathbb{R}^n} \psi(r)\bar s_r + \lambda\bar z = \psi(\bar r) + \lambda\bar x_{n+1} = \max_{i=1,\ldots,t} a_i\bar r + \lambda\bar x_{n+1}$$
$$= \max_{i=1,\ldots,t} [a_i(\bar x - f) + (\lambda - a_i d)\bar x_{n+1}].$$

Thus there exists $i \in \{1, \ldots, t\}$ such that $a_i(\bar x - f) + (\lambda - a_i d)\bar x_{n+1} \geqslant 1$. This shows that $\binom{\bar x}{\bar x_{n+1}}$ is not in the interior of $B(\lambda)$. $\square$

Theorem 4 implies that $\pi_l(d)$ is the minimum value of λ such that $B(\lambda)$ is $(S \times \mathbb{Z}_+)$-free.

Example (Continued). In the previous example, let $d \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. If $\lambda \neq 0$, then the set $B(\lambda)$ is the two-dimensional polyhedron with two facets, containing the points $\binom{0}{0}$ and $\binom{1}{0}$, respectively, and with one vertex, namely $\binom{f}{0} + \lambda^{-1}\binom{d}{1}$. If $\lambda = 0$, then $B(\lambda)$ is the split set $[0, 1] + \langle\binom{d}{1}\rangle$. It is immediate to verify that for $\lambda < 0$, the interior of $B(\lambda)$ contains one of the integral points $\binom{\lfloor d\rfloor}{1}$ or $\binom{\lceil d\rceil}{1}$.

For example, let $f = \frac{1}{4}$. See Figure 1. For $d = \frac{3}{2}$, $\psi_B(d) = 2$. One can readily verify that $B(\lambda)$ is $\mathbb{Z} \times \mathbb{Z}_+$-free if and only if $\lambda \geqslant \frac{2}{3}$; otherwise, it contains the point $\binom{2}{1}$. Hence $\pi_l(d) = \frac{2}{3}$.

For $d = 1$, $\psi_B(d) = \frac{4}{3}$. It is immediate that $B(\lambda)$ is $\mathbb{Z} \times \mathbb{Z}_+$-free if and only if $\lambda \geqslant 0$, hence $\pi_l(d) = 0$.

Theorem 5. *Let ψ be a minimal valid function for (3) and π be a minimal lifting of ψ. Then there exists $\varepsilon > 0$ such that ψ, π, and π_l coincide on the ball of radius ε centered at the origin.*

Proof. Because ψ is a minimal valid function for (3), there exists a maximal S-free convex set $B = \{x \in \mathbb{R}^n \mid a_i \cdot (x-f) \leqslant 1,\ i = 1, \ldots, t\}$ such that $\psi = \psi_B$.

Let

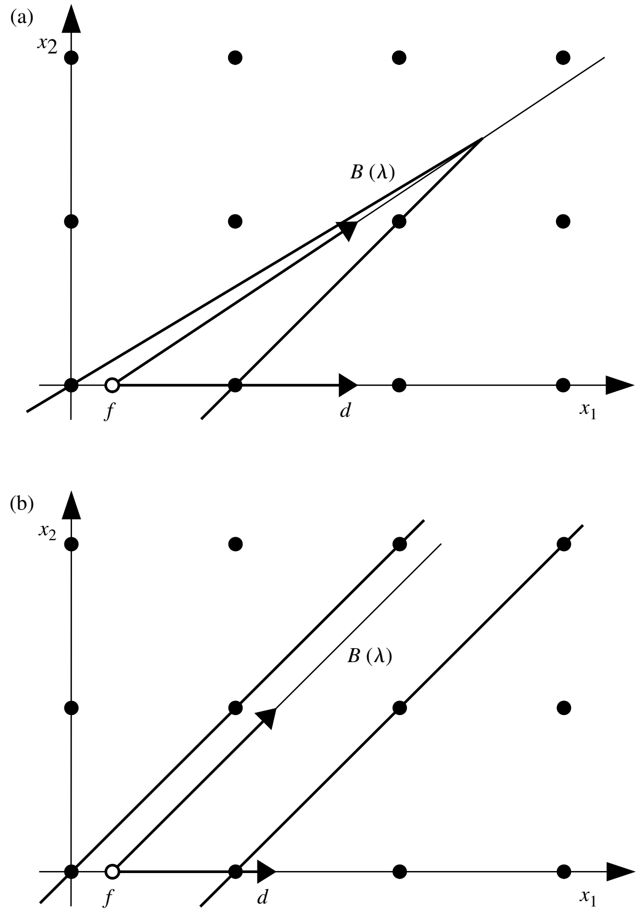
$$\alpha = \max_{1\leqslant i,\, j\leqslant t}\ \max_{\|r\|=1} (a_i - a_j)r.$$

Because B is a maximal S-free convex set, every facet of B contains a point of S in its relative interior. Hence, for $i = 1, \ldots, t$, there exists $x^i \in S$ such that $a_i(x^i - f) = 1$ and $a_j(x^i - f) \leqslant 1 - \gamma_i$, $j \neq i$, for some positive γ_i. Let $\varepsilon > 0$ such that $\varepsilon\alpha \leqslant \gamma_i$ for $i = 1, \ldots, t$.

Let $d \in \mathbb{R}^n$ such that $\|d\| \leqslant \varepsilon$. We will show that for every $\lambda < \psi(d)$, $B(\lambda)$ contains a point of $S \times \mathbb{Z}_+$ in its interior. By Theorem 4, this implies that $\pi_l(d) \geqslant \psi(d)$. Because $\pi_l \leqslant \pi \leqslant \psi$, this implies $\pi_l(d) = \pi(d) = \psi(d)$.

Let i, $1 \leqslant i \leqslant t$, such that $\psi(d) = a_i d$. Let $\lambda = \psi(d) - \delta$ for some $\delta > 0$. We show that $B(\lambda)$ contains the point $\binom{x^i}{1}$

Figure 1. Example: $f=\frac{1}{4}$. Top: $d=\frac{3}{2}$. Bottom: $d=1$.



in its interior. Indeed, by (7), $B(\lambda)$ is the set of points in $\mathbb{R}^{n+1}$ satisfying the inequalities

$$a_j(x-f)+[(a_i-a_j)d-\delta]x_{n+1}\leqslant 1,\quad j=1,\ldots,t.$$

Substituting $\binom{x^i}{1}$, we obtain

$$a_i(x^i-f)-\delta<1,$$
$$a_j(x^i-f)+(a_i-a_j)d-\delta<1,\quad j=1,\ldots,t,\ j\neq i,$$

where the first inequality follows from $a_i(x^i-f)=1$, while the second follows from $a_j(x^i-f)\leqslant 1-\gamma_i$, $\|d\|\leqslant\varepsilon$, and $(a_i-a_j)(d/\|d\|)\leqslant\alpha$ by our choice of α.

Thus $\binom{x^i}{1}$ is in the interior of $B(\lambda)$. □

Example (Continued). From the previous example where $n=1$, $0<f<1$ and $S=\mathbb{Z}$, note that $\pi_l(d)=\psi_B(d)$ for every $d\in[-f,1-f]$. Indeed, if $d<0$, then $B(\lambda)$ contains $\binom{0}{1}$ for all $\lambda<\psi_B(d)$; while if $d\geqslant 0$, then $B(\lambda)$ contains $\binom{1}{1}$ for all $\lambda<\psi_B(d)$. Furthermore, for $\lambda=\psi_B(d)$, if $d<0$ the facet of $B(\lambda)$ containing $\binom{0}{0}$ is vertical and contains the point $\binom{0}{1}$, if $d\geqslant 0$, then the facet of $B(\lambda)$ containing $\binom{1}{0}$ is vertical and contains the point $\binom{1}{1}$.

Theorem 5 implies that, for every minimal valid function ψ for (3), there exists a region $R_\psi\subseteq\mathbb{R}^n$ containing the origin in its interior such that ψ and π coincide in R_ψ for every minimal lifting π of ψ for (1). Because ψ is piecewise linear, it follows that π is piecewise linear around the origin. This is in contrast with extreme functions π for the pure integer semi-infinite relaxation (i.e., the set (1) where all the s_r are set to 0), which need not be piecewise linear (Basu et al. 2010a).

Lemma 6. *Let ψ be a minimal valid function, and π be a minimal lifting of ψ. Then*

(i) *for every $r\in\mathbb{R}^n$ and $w\in\mathbb{Z}^n\cap\operatorname{lin}(\operatorname{conv}(S))$, $\pi(r)=\pi(r+w)$;*

(ii) *for every $r\in\mathbb{R}^n$ such that $r+w\in R_\psi$ for some $w\in\mathbb{Z}^n\cap\operatorname{lin}(\operatorname{conv}(S))$, $\pi(r)=\psi(r+w)$.*

Proof. (i) Let $\bar r\in\mathbb{R}^n$ and $w\in\mathbb{Z}^n\cap\operatorname{lin}(\operatorname{conv}(S))$. Suppose $\pi(\bar r)\neq\pi(\bar r+w)$. Because $-w\in\mathbb{Z}^n\cap\operatorname{lin}(\operatorname{conv}(S))$, we may assume $\pi(\bar r)>\pi(\bar r+w)$. Because $w\in\mathbb{Z}^n\cap\operatorname{lin}(\operatorname{conv}(S))$, then a point $x\in\mathbb{R}^n$ is in S if and only if $x+w\in S$. Thus a point $(\bar x,\bar s,\bar y)$ satisfies (1) if and only if $(\bar x+w\bar y_{\bar r},\bar s,\tilde y)$ satisfies (1), where $\tilde y_{\bar r}=0$, $\tilde y_{\bar r+w}=\bar y_{\bar r+w}+\bar y_{\bar r}$, and $\tilde y_r=\bar y_r$ for every $r\in\mathbb{R}^n\setminus\{\bar r,\bar r+w\}$. This shows that the function π' defined by $\pi'(\bar r)=\pi(\bar r+w)$, $\pi'(r)=\pi(r)$ for every $r\in\mathbb{R}^n\setminus\{\bar r\}$ is a lifting of ψ, contradicting the minimality of π.

(ii) It follows from (i) that $\pi(r)=\pi(r+w)$. By definition of R_ψ, $\pi(r+w)=\psi(r+w)$. □

This lemma is closely related to a result of Balas and Jeroslow (1980). It implies the following property.

Theorem 7. *If for every $r\in\mathbb{R}^n$ there exists $w^r\in\mathbb{Z}^n\cap\operatorname{lin}(\operatorname{conv}(S))$ such that $r+w\in R_\psi$, then there exists a unique minimal lifting of ψ, namely the function π defined by $\pi(r)=\psi(r+w^r)$. Furthermore, $\pi=\pi_l$.*

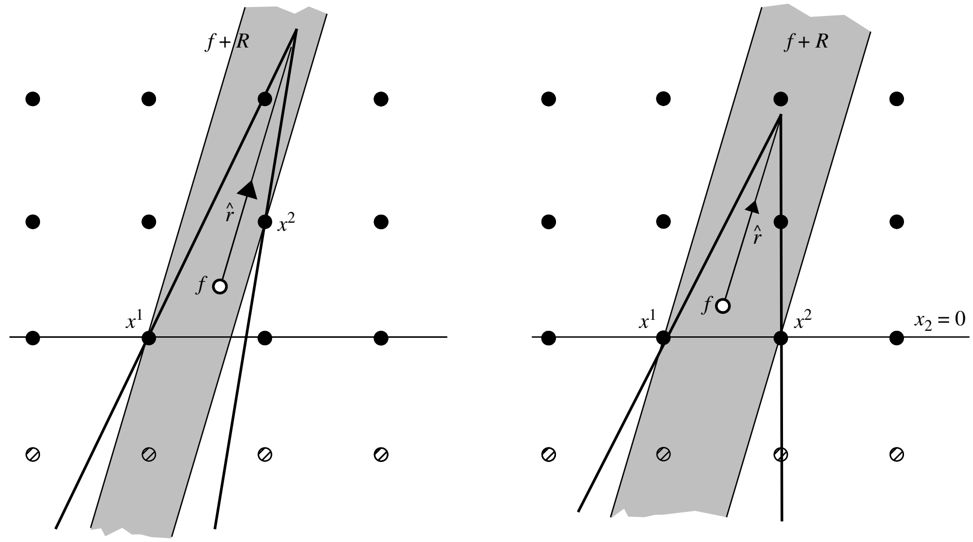
Example (Continued). From the previous example where $n=1$, $0<f<1$, and $S=\mathbb{Z}$, we have shown that $\psi(r)=\pi_l(r)$ for every $r\in[-f,1-f]$. Note that, for every $r\in\mathbb{R}$, $r-\lfloor r+f\rfloor\in[-f,1-f]$. Thus $\pi_l(r)=\psi(r-\lfloor r+f\rfloor)$ for all $r\in\mathbb{R}$, and π_l is the unique minimal lifting of ψ. Thus $\pi_l(r)=\max\{-f^{-1}(r-\lfloor r+f\rfloor),(1-f)^{-1}(r-\lfloor r+f\rfloor)\}$. More explicitly, if $r-\lfloor r\rfloor<1-f$, then $\pi_l(r)=(r-\lfloor r\rfloor)/(1-f)$, while if $r-\lfloor r\rfloor\geqslant 1-f$, $\pi_l(r)=(\lceil r\rceil-r)/f$.

Given a tableau row $x=f+\sum_{i=1}^h p^i s_i+\sum_{j=1}^k q^j y_j$, where $s_i\geqslant 0$, $i=1,\ldots,h$, and $y_j\geqslant 0$ and integer, $j=1,\ldots,h$, the inequality $\sum_{i=1}^h\psi(p^i)s_i+\sum_{j=1}^k\pi_l(q^j)y_j\geqslant 1$ is

$$\sum_{\substack{i=1\\p^i\geqslant 0}}^{h}\frac{p^i}{1-f}s_i+\sum_{\substack{i=1\\p^i<0}}^{h}-\frac{p^i}{f}s_i+\sum_{\substack{j=1\\q^j-\lfloor q^j\rfloor<1-f}}^{k}\frac{q^j-\lfloor q^j\rfloor}{1-f}y_j$$
$$+\sum_{\substack{j=1\\q^j-\lfloor q^j\rfloor\geqslant 1-f}}^{k}\frac{\lceil q^j\rceil-q^j}{f}y_j\geqslant 1,$$

which is the Gomory mixed-integer cut associated with the tableau row.

Figure 2. Wedges and corresponding region $R+\{f\}$ shaded in gray.



Note. The inequality corresponding to the wedge on the right has a unique minimal lifting.

3. Applications

We illustrated in §2 how our geometric approach can be used to derive Gomory's mixed-integer cuts. In this section, we give three examples of how it can be applied to the multirow case.

3.1. Wedge Inequalities

We consider the problem (1) where $n=2$ and $S=\mathbb{Z}\times\mathbb{Z}_+$. We focus on inequalities arising from maximal S-free convex sets with two sides and one vertex. We call such sets *wedges*.

Let $B=\{x\in\mathbb{R}^2 \mid a_i(x-f)\leqslant 1,\ i=1,2\}$ be such a maximal S-free convex set. Because B is S-free, its only vertex must be in the interior of $\operatorname{conv}(S)$, $\operatorname{rec}(B)$ has dimension 2, and for every nonzero element $r\in\operatorname{rec}(B)$, $r_2<0$.

Note that $\operatorname{rec}(\operatorname{conv}(S))=\mathbb{R}\times\mathbb{R}_+$ and B has empty lineality space. By Theorem 3, $\operatorname{lin}(B)\supseteq\operatorname{rec}(B\cap\operatorname{conv}(S))$, hence $\operatorname{rec}(B)\cap\operatorname{conv}(S)=\varnothing$. In particular, $(\mathbb{R}\times\{0\})\cap\operatorname{rec}(B)=\varnothing$, thus by symmetry we may assume $a_1\binom{1}{0}<0$ and $a_2\binom{1}{0}>0$, that is, $a_{11}<0$ and $a_{21}>0$.

Let $\hat r$ be a nonzero vector such that $a_1\hat r=a_2\hat r$. Clearly, the second coordinate of $\hat r$ is nonzero. Note that any point $x\in\mathbb{R}^2$ can be uniquely written as $x=f+\alpha^x\hat r+\beta^x\binom{1}{0}$ where $\alpha^x,\beta^x\in\mathbb{R}$. Let $\bar x\in S$ be a point in the relative interior of one of the two facets of B, say $a_h(\bar x-f)=1$, $a_k(\bar x-f)<1$. Note that $0>(a_k-a_h)(\bar x-f)=\beta^{\bar x}(a_{k1}-a_{h1})$, hence $\beta^{\bar x}<0$ if $h=1$ and $\beta^{\bar x}>0$ if $h=2$. Let x^1 be a point of S in the relative interior of the facet defined by $a_1(x-f)\leqslant 1$ such that β^{x^1} is largest possible, and let x^2 be a point of S in the relative interior of the facet defined by $a_2(x-f)\leqslant 1$ such that β^{x^2} is smallest possible. Let $\beta_i=\beta^{x^i}$. Note that $\beta_1<0<\beta_2$. We define the region $R=[\beta_1,\beta_2]+\langle\hat r\rangle$. (See Figure 2.)

Lemma 8. *For every $d\in R$, $\pi_l(d)=\psi_B(d)$.*

Proof. Let $d\in R$, that is $d=\alpha\hat r+\beta\binom{1}{0}$, for some $\alpha\in\mathbb{R}$ and $\beta\in[\beta_1,\beta_2]$. We consider the case $\beta\leqslant 0$. The case $\beta\geqslant 0$ is similar.

Note that $(a_1-a_2)d=\alpha(a_1-a_2)\hat r+\beta(a_{11}-a_{21})\geqslant 0$ because $(a_1-a_2)\hat r=0$, $\beta\leqslant 0$, $a_{11}<0$, and $a_{21}>0$. Hence $\psi_B(d)=\max\{a_1d,a_2d\}=a_1d$.

We will show that, for every $\lambda<\psi_B(d)$, the set $B(\lambda)$ defined in (7) contains the point $\binom{x^1}{1}$ in its interior. By Theorem 4, this will imply $\pi_l(d)\geqslant\psi_B(d)$, and thus $\pi_l(d)=\psi_B(d)$.

Let $\lambda=\psi_B(d)-\delta$ for some $\delta>0$. Then $B(\lambda)$ is the set of $x\in\mathbb{R}^3$ satisfying

$$a_1(x-f)-\delta x_3\leqslant 1,$$
$$a_2(x-f)+(a_1-a_2)dx_3-\delta x_3\leqslant 1.$$

Substituting $\binom{x^1}{1}$ in the first inequality, we obtain $a_1(x^1-f)-\delta=1-\delta<1$. Substituting in the second inequality, we obtain

$$\begin{aligned}&a_2(x^1-f)+(a_1-a_2)d-\delta\\&\quad=\alpha^{x^1}a_2\hat r+\beta_1a_{21}+\alpha(a_1-a_2)\hat r+\beta(a_{11}-a_{21})-\delta\\&\quad=\alpha^{x^1}a_1\hat r+\beta_1a_{11}+(\beta-\beta_1)(a_{11}-a_{21})-\delta\\&\quad\leqslant a_1(x^1-f)-\delta=1-\delta<1,\end{aligned}$$

where the first inequality in the last row follows from $\beta_1\leqslant\beta$, $a_{11}<0$, $a_{21}>0$. Thus $\binom{x^1}{1}$ is in the interior of $B(\lambda)$. □

Let y^1 and y^2 be the intersection of the facets defined by $a_1(x-f)\leqslant 1$ and $a_2(x-f)\leqslant 1$, respectively, with the axis $x_2=0$. That is, $a_1(y^1-f)=1$, $y^1_2=0$, and $a_2(y^2-f)=1$, $y^2_2=0$. Because B is S-free, $y^2_1-y^1_1\leqslant 1$, where equality

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holds if and only if y^1, y^2 are integral. Furthermore, it is not difficult to show that $\beta_2 - \beta_1 \leqslant y_1^2 - y_1^1$. Thus $\beta_2 - \beta_1 = 1$ if and only if y^1, y^2 are integral vectors. In this case, for every $r \in \mathbb{R}^2$ there exists $w^r \in \mathbb{Z} \times \{0\}$ such that $r + w^r \in R$. Because $\text{lin}(\text{conv}(S)) = \mathbb{R} \times \{0\}$, by Theorem 7, $\pi_l(r)$ is the unique minimal lifting of ψ_B, and $\pi_l(r) = \psi_B(r + w^r)$ for every $r \in \mathbb{R}^2$.

Dey and Wolsey (2010) show that ψ_B is extreme for (3) if and only if B contains at least three points of S. Thus Remark 2 implies the following theorem.

Theorem 9. *If B contains at least three points of S and $B \cap (\mathbb{R} \times \{0\})$ is an interval of length* 1, *then (ψ_B, π_l) is a valid extreme inequality for* (1).

Example. Let $f = \binom{2/3}{1/3}$ and $S = \mathbb{Z} \times \mathbb{Z}_+$. Consider the wedge

$$W = \Big\{x \in \mathbb{R}^2 \mid -3\big(x_1 - \tfrac{2}{3}\big) + 3\big(x_2 - \tfrac{1}{3}\big) \leqslant 1, \\ \tfrac{12}{5}\big(x_1 - \tfrac{2}{3}\big) - \tfrac{3}{5}\big(x_2 - \tfrac{1}{3}\big) \leqslant 1\Big\}.$$

The set W is a maximal S-free convex set, as one may easily see from Figure 3.

The corresponding minimal inequality is given by

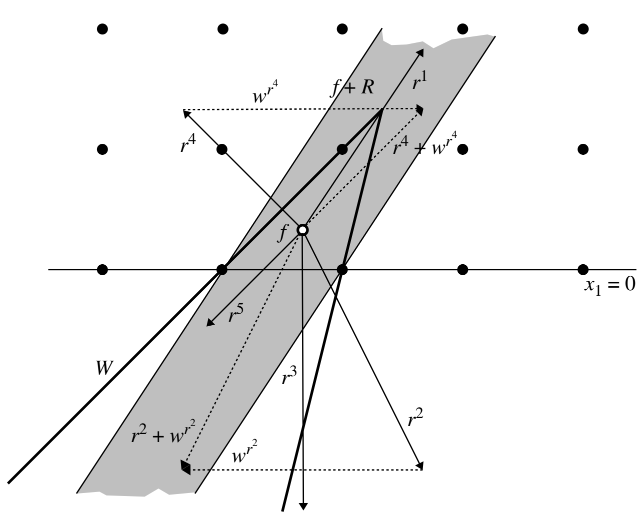
$$\psi(r) = \max\big\{-3r_1 + 3r_2,\ \tfrac{12}{5}r_1 - \tfrac{3}{5}r_2\big\}.$$

One can easily verify that the vector $\hat{r} = \binom{2}{3}$ satisfies $-3\hat{r}_1 + 3\hat{r}_2 = \frac{12}{5}\hat{r}_1 - \frac{3}{5}\hat{r}_2$, and that the region R is thus given by $R = [-\frac{4}{9}, \frac{5}{9}] + \langle \binom{2}{3} \rangle$. This can be written as

$$R = \big\{r \in \mathbb{R}^2 \mid -\tfrac{4}{9} \leqslant r_1 - \tfrac{2}{3}r_2 \leqslant \tfrac{5}{9}\big\}.$$

For every $r \in R^2$, define the integral vector w^r by $w_1^r = -\lfloor r_1 - \frac{2}{3}r_2 - \frac{4}{9}\rfloor$, $w_2^r = 0$. Note that $w^r \in \text{lin}(S) \cap \mathbb{Z}^2$ and $r + w^r \in R$ for all $r \in \mathbb{R}^2$. The unique minimal lifting for ψ

Figure 3. Set W in the example and corresponding region $R + \{f\}$ shaded in gray.



is therefore the function π defined by $\pi(r) = \psi(r + w^r)$. The explicit formula is given by

$$\pi(r) = \max\Big\{-3\big(r_1 - \lfloor r_1 - \tfrac{2}{3}r_2 - \tfrac{4}{9}\rfloor\big) + 3r_2, \\ \tfrac{12}{5}\big(r_1 - \lfloor r_1 - \tfrac{2}{3}r_2 - \tfrac{4}{9}\rfloor\big) - \tfrac{3}{5}r_2\Big\}.$$

Suppose now that we are given the following two rows of the optimal simplex tableau for the linear relaxation of a mixed integer program:

$$x_1 = \tfrac{2}{3} + x_3 + x_4 - x_6 - \tfrac{4}{5}x_7,$$
$$x_2 = \tfrac{1}{3} + \tfrac{3}{2}x_3 - 2x_4 - \tfrac{7}{3}x_5 + x_6 - \tfrac{4}{5}x_7,$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geqslant 0,$$
$$x_1, x_2, x_4, x_6 \in \mathbb{Z}.$$

The lifted inequality determined by the wedge W is $\psi(r^1)x_3 + \pi(r^2)x_4 + \psi(r^3)x_5 + \pi(r^4)x_6 + \psi(r^5)x_7 \geqslant 1$, where $r^1 = \binom{1}{3/2}$, $r^2 = \binom{1}{-2}$, $r^3 = \binom{0}{-7/3}$, $r^4 = \binom{-1}{1}$, $r^5 = \binom{-4/5}{-4/5}$. This gives the inequality

$$\tfrac{3}{2}x_3 - \tfrac{6}{5}x_4 + \tfrac{7}{5}x_5 + \tfrac{9}{5}x_6 \geqslant 1.$$

Note that the nonlifted inequality (that is, the inequality obtained from W if we ignored the integrality conditions on x_4 and x_6) is

$$\tfrac{3}{2}x_3 + \tfrac{18}{5}x_4 + \tfrac{7}{5}x_5 + 6x_6 \geqslant 1.$$

3.2. Simplicial Polytopes

In this section we focus on valid inequalities for (3) arising from maximal lattice-free simplicial polytopes, in the case where $S = \mathbb{Z}^n$. Recall that a polytope is *simplicial* if each of its facets is a simplex.

Let $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leqslant 1,\ i = 1, \ldots, t\}$ be an n-dimensional maximal lattice-free simplicial polytope, and let $v^1, \ldots, v^p$ be its vertices. For $i = 1, \ldots, t$, let $V_i \subset \{1, \ldots, p\}$ be the set of indices of vertices of the facet defined by $a_i(x - f) \leqslant 1$, that is $a_i(v^j - f) = 1$ for all $j \in V_i$. Let $r^i = v^i - f$, $i = 1, \ldots, p$. Note that because B is simplicial, $\{r^j \mid j \in V_i\}$ consists of n linearly independent vectors, for $i = 1, \ldots, t$, and $a_i r^j = 1$ for all $j \in V_i$, while $a_i r^j < 1$ for all $j \notin V_i$.

Let $\bar{x}$ be an integral point in the relative interior of the facet defined by $a_i(x - f) \leqslant 1$, that is, $a_i(\bar{x} - f) = 1$, $a_j(\bar{x} - f) < 1$, $j \neq i$. Then $\bar{x}$ can be uniquely written as $\bar{x} = f + \sum_{j \in V_i} \bar{\alpha}_j r^j$, where $\sum_{j \in V_i} \bar{\alpha}_j = 1$, $\bar{\alpha}_j \geqslant 0$, $j \in V_i$. Let $R(\bar{x}) = \{\sum_{j \in V_i} \alpha_j r^j \mid 0 \leqslant \alpha_j \leqslant \bar{\alpha}_j,\ j \in V_i\}$.

Let us denote by $\mathcal{I}$ the set of all points $\bar{x}$ in $\mathbb{Z}^n$ such that $\bar{x}$ is contained in the relative interior of some facet of B. Let $R = \bigcup_{\bar{x} \in \mathcal{I}} R(\bar{x})$.

Lemma 10. *For every $d \in R$, $\pi_l(d) = \psi_B(d)$.*

Proof. We need only to show that, given $\bar{x} \in \mathcal{I}$ and $d \in R(\bar{x})$, $\pi_l(d) = \psi_B(d)$. By symmetry we may assume that $\bar{x}$ is in the relative interior of the facet defined by $a_1(\bar{x} - f) \leqslant 1$, and that $V_1 = \{1, \ldots, n\}$. Let $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ nonnegative such that $\sum_{j=1}^n \bar{\alpha}_j = 1$ and $\bar{x} = f + \sum_{j=1}^n \bar{\alpha}_j r^j$. Because $d \in R(\bar{x})$, there exist $\alpha_1, \ldots, \alpha_n$ such that $d = \sum_{j=1}^n \alpha_j r^j$ and $0 \leqslant \alpha_j \leqslant \bar{\alpha}_j$, $j = 1, \ldots, n$. Note that, for $i = 1, \ldots, t$, $(a_1 - a_i)d = \sum_{j=1}^n \alpha_j (a_1 - a_i) r^j \geqslant 0$. Thus $\psi_B(d) = a_1 d$.

We will show that for every $\lambda < \psi_B(d)$, the set $B(\lambda)$ defined as in (7) contains the point $\binom{\bar{x}}{1}$ in its interior. By Theorem 4, this will imply $\pi_l(d) \geqslant \psi_B(d)$, and thus $\pi_l(d) = \psi_B(d)$.

Let $\lambda = \psi_B(d) - \delta$ for some $\delta > 0$. Then $B(\lambda)$ is the set of $x \in \mathbb{R}^{n+1}$ satisfying

$$a_1(x - f) - \delta x_{n+1} \leqslant 1,$$

$$a_i(x - f) + (a_1 - a_i) d x_{n+1} - \delta x_{n+1} \leqslant 1, \quad i = 2, \ldots, t.$$

Substituting $\binom{\bar{x}}{1}$ in the first inequality, we obtain $a_1(\bar{x} - f) - \delta = 1 - \delta < 1$. Substituting in the ith inequality, $i = 2, \ldots, n+1$, we obtain

$$\begin{aligned} & a_i(\bar{x} - f) + (a_1 - a_i)d - \delta \\ & = \sum_{j=1}^n \bar{\alpha}_j a_i r^j + \sum_{j=1}^n \alpha_j (a_1 - a_i) r^j - \delta \\ & = \sum_{j=1}^n \bar{\alpha}_j - \sum_{j=1}^n \bar{\alpha}_j (1 - a_i r^j) + \sum_{j=1}^n \alpha_j (1 - a_i r^j) - \delta \\ & = 1 - \sum_{j=1}^n (\bar{\alpha}_j - \alpha_j)(1 - a_i r^j) - \delta \\ & \leqslant 1 - \delta < 1, \end{aligned}$$

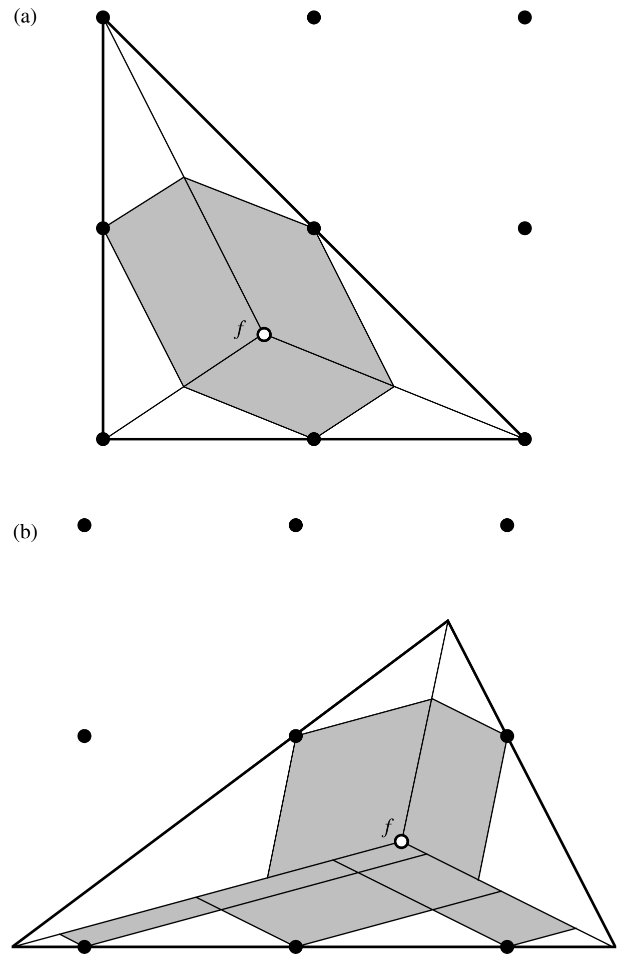
where the equality in the third line follows from $a_i r^j = 1$ for $j = 1, \ldots, n$, the equality on the fourth line follows from $\sum_{j=1}^n \bar{\alpha}_j = 1$, while the first inequality on the last line follows from $\alpha_j \leqslant \bar{\alpha}_j$ and $a_i r^j \leqslant 1$. □

In light of Theorem 7, we are interested in cases where for every $r \in \mathbb{R}^n$ there exists $w^r \in \mathbb{Z}^n$ such that $r + w^r \in R$, because in this case π_l is the unique minimal lifting.

Dey and Wolsey (2008) studied the case $n = 2$. In this case maximal lattice-free polytopes are either triangles or quadrilaterals (Lovász 1989). Dey and Wolsey show that the above property holds if and only if B is a triangle containing at least four integral points (see Figure 4), while it does not hold if B is a triangle containing exactly three integral points or if B is a quadrilateral. They also show that when B is a triangle with at least four integral points, (ψ_B, π_l) is extreme for (1). This fact also follows from Remark 2 and from the fact that ψ_B is extreme for (3) whenever B is a maximal lattice-free triangle (Cornuéjols and Margot 2009).

We next show that the above property holds when B is the n-dimensional simplex $\text{conv}\{0, ne_1, \ldots, ne_n\}$, where e_i denotes the ith unit vector. We assume that f is in the interior of B. The picture on the top in Figure 4 shows the case $n = 2$. Note that $B = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leqslant n,\ x_i \geqslant 0,\ i = 1, \ldots, n\}$. The point $e - e_i$, where e denotes the vector of all ones, is the unique integral point in the relative interior of the facet of B defined by $x_i \geqslant 0$, and e is the unique integral point in the relative interior of the facet of B defined by $\sum_{i=1}^n x_i \leqslant n$. Thus $\mathcal{I} = \{e, e - e_1, \ldots, e - e_n\}$.

Figure 4. Lattice-free triangles giving inequalities with a unique minimal lifting.



Note. Region $R + \{f\}$ is shaded.

Let $d^1, \ldots, d^{n+1}$ be defined as follows: $d^i = e_i - (1/n)f$, $i = 1, \ldots, n$ and $d^{n+1} = -(1/n)f$. Then $R(e) = \{\sum_{j=1}^n \alpha_j d^j \mid 0 \leqslant \alpha_i \leqslant 1,\ i = 1, \ldots, n\}$ and $R(e - e_i) = \{\sum_{j=1}^{n+1} \alpha_j d^j \mid 0 \leqslant \alpha_k \leqslant 1,\ k = 1, \ldots, n+1,\ \alpha_i = 0\}$. Therefore, $R = \{\sum_{j=1}^{n+1} \alpha_j d^j \mid 0 \leqslant \alpha_i \leqslant 1,\ i = 1, \ldots, n+1,\ \alpha_i = 0$ for some $i,\ 1 \leqslant i \leqslant n+1\}$.

Lemma 11. *Let $B = \text{conv}\{0, ne_1, \ldots, ne_n\}$. For every $r \in \mathbb{R}^n$, there exists $w \in \mathbb{Z}^n$ such that $r + w \in R$.*

Proof. Note that for $1 \leqslant i, j \leqslant n+1$, $d^i - d^j \in \mathbb{Z}^n$.

Let $C_i = \text{cone}\{d^j \mid j \neq i, 1 \leqslant j \leqslant n+1\}$, $i = 1, \ldots, n+1$. Note that $\bigcup_{i=1}^{n+1} C_i = \mathbb{R}^n$ and $C_i \cap C_k = \text{cone}\{d^j \mid$

$j \neq i, k$, $1 \leqslant j \leqslant n+1\}$. Furthermore, $-d^i \in C_i$ for $i = 1, \dots, n+1$.

Claim. *Let $r \in \mathbb{R}^n$ and let i such that $r \in C_i$. There exists a unique $\alpha \in \mathbb{R}^{n+1}$ such that $r = \sum_{j=1}^{n+1} \alpha_j d^j$ and $\alpha_i = 0$. Furthermore, α is nonnegative and $\alpha_j \leqslant \alpha'_j$ for every nonnegative $\alpha' \in \mathbb{R}^{n+1}$ such that $r = \sum_{j=1}^{n+1} \alpha'_j d^j$.*

We prove the claim. Because C_i is generated by n linearly independent vectors, r can be uniquely written as $r = \sum_{j=1}^{n+1} \alpha_j d^j$ such that $\alpha_i = 0$, and α must be nonnegative because $r \in C_i$. Given a nonnegative $\alpha' \in \mathbb{R}^{n+1}$ such that $r = \sum_{j=1}^{n+1} \alpha'_j d^j$ distinct from α, then $\alpha'_i > 0$. Hence,

$$-d^i = (\alpha'_i)^{-1} \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (\alpha'_j - \alpha_j) d^j;$$

thus $\alpha'_j - \alpha_j \geqslant 0$ because $-d^i \in C_i$. Hence, by the above argument $-d^i$ can be uniquely written as a linear combination of the extreme rays of C_i, and such combination is nonnegative. This proves the claim.

Let us now consider $r \in \mathbb{R}^n$. Let i be such that $r \in C_i$, $1 \leqslant i \leqslant n+1$. Let $\alpha \in \mathbb{R}^{n+1}$ such that $r = \sum_{j=1}^{n+1} \alpha_j d^j$ and $\alpha_i = 0$. By the above claim α is nonnegative. Let $\bar{\alpha} = \max_{j=1,\dots,n+1} \alpha_j$. If $\bar{\alpha} \leqslant 1$, then $r \in R$. If not, $\alpha_k = \bar{\alpha} > 1$ for some $1 \leqslant k \leqslant n+1$.

Let $r' = r + (e_i - e_k) = r + (d^i - d^k)$. Then $r' = \sum_{j \neq i,k} \alpha_j d^j + d^i + (\alpha_k - 1) d^k$. Let h be such that $r' \in C_h$, $1 \leqslant h \leqslant n+1$ and let $\alpha' \in \mathbb{R}^{n+1}$ be the unique vector such that $r' = \sum_{j=1}^{n+1} \alpha'_j d^j$ and $\alpha'_h = 0$. By the previous claim, α' satisfies the following properties:

- $r' - r \in \mathbb{Z}^n$ and $\alpha'_h = 0$,
- $0 \leqslant \alpha'_j \leqslant \alpha_j$, $j \neq i$, $1 \leqslant j \leqslant n+1$,
- $0 \leqslant \alpha'_i \leqslant 1$, $0 \leqslant \alpha'_k \leqslant \alpha_k - 1$.

Thus, either $\max_{j=1,\dots,n+1} \alpha'_j \leqslant \bar{\alpha} - 1$, or the number of indices j such that $\alpha'_j = \bar{\alpha}$ is smaller than the number of indices j such that $\alpha_j = \bar{\alpha}$. This implies the statement of the lemma. ☐

It can be shown that, in this case, R is a polytope with $\binom{n+1}{2}$ pairs of parallel facets, and that R has volume 1. Thus, by Lemma 11, all possible translations of R by integral vectors form a tiling of $\mathbb{R}^n$. Therefore for every $d \in \mathbb{R}^n$, there exists $w^d \in \mathbb{Z}^n$ such that $d + w^d \in R$. By Theorem 7, the function π_l defined by $\pi_l(d) = \psi_B(d + w^d)$ is the unique minimal lifting of ψ_B.

Whenever B is a maximal lattice-free simplex, ψ_B is extreme for (3). Indeed, if $v^1, \dots, v^{n+1}$ are the vertices of B and we define $r^j = v^j - f$, $j = 1, \dots, n+1$, ψ_B is extreme for (3) if and only if $\sum_{j=1}^{n+1} \psi_B(r^j) s_j \geqslant 1$ is extreme for the convex hull of the set $R_f(r^1, \dots, r^{n+1})$ defined as the set of all $s \in \mathbb{R}^{n+1}$ such that $f + \sum_{j=1}^{n+1} r^j s_j \in \mathbb{Z}^n$ and $s \geqslant 0$ (see Dey and Wolsey 2010). In this case, because each facet of B contains an integral point, for $i = 1, \dots, n+1$ there exists $s^i \in \mathbb{R}^{n+1}$ such that $s^i_j > 0$ for all $j \neq i$, $1 \leqslant j \leqslant n+1$, $s^i_i = 0$ and $\sum_{j=1}^{n+1} s^i_j r^j \in \mathbb{Z}^n$. Hence $s^1, \dots, s^{n+1}$ are linearly independent points of $R_f(r^1, \dots, r^{n+1})$, and $\sum_{j=1}^{n+1} \psi_B(r^j) s^i_j = 1$ for $i = 1, \dots, n+1$. This shows that $\sum_{j=1}^{n+1} \psi_B(r^j) s_j \geqslant 1$ defines a facet of $\text{conv}(R_f(r^1, \dots, r^{n+1}))$, and thus it is extreme for $\text{conv}(R_f(r^1, \dots, r^{n+1}))$. Therefore, ψ_B is extreme for (3).

The above statement and Remark 2 imply the following.

Theorem 12. *If $B = \text{conv}(0, ne_1, \dots, ne_n)$, (ψ_B, π_l) is extreme for (1) with $S = \mathbb{Z}^n$.*

By standard arguments, the above theorem holds up to unimodular transformations and integer translations of the set B. Namely, given any unimodular $n \times n$-matrix U (i.e., an integral matrix with determinant one) and any vector $v \in \mathbb{Z}^n$ such that f is in the interior of the set $B' = \text{conv}(v, n(Ue_1) + v, \dots, n(Ue_n) + v)$, then $(\psi_{B'}, \pi_l)$ is extreme for (1) with $S = \mathbb{Z}^n$. Note that, given a vector $f \notin \mathbb{Z}^n$, one can always find an appropriate unimodular matrix U and integral vector v so that f is in the interior of the corresponding set B'. This type of lifted inequality has been used in computational experiments by Espinoza (2010) and recently by Balas and Qualizza (2009), and the results seem to indicate that such cuts might be useful in practice.

3.3. Simple Cones

We consider the case where $S = \mathbb{Z}^{n-1} \times \mathbb{Z}_+$ and the maximal S-free convex set B is the translation of a simple cone. That is, B has a unique vertex v, and $B - v$ is a simple cone. Recall that a polyhedral cone in $\mathbb{R}^n$ is *simple* if it is generated by n linearly independent vectors, and therefore it has n facets. This case extends the wedge inequalities of §3.1.

Let $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leqslant 1,\ i = 1 \dots, n\}$. By Theorem 3, $\text{rec}(B \cap \text{conv}(S))$ is contained in the lineality space of B, which is empty. Therefore, $B \cap \text{conv}(S)$ is bounded. Therefore the polytope $B \cap (\mathbb{R}^{n-1} \times \{0\})$ is an an $(n-1)$-dimensional simplex P. Let $v^1, \dots, v^n$ be the vertices of P, and let $r^j = v^j - f$, $j = 1, \dots, n$. By symmetry, we may assume that $a_i r^j = 1$ for $1 \leqslant i, j \leqslant n$, $i \neq j$, and $a_i r^i < 1$. Let $\hat{r} = v - f$. Note that for $i = 1, \dots, n$, $a_i \hat{r} = 1$.

Let $\bar{x}$ be a point of S in the relative interior of one of the facets of B, say the facet defined by $a_h(x - f) \leqslant 1$. Then $\bar{x}$ can be uniquely written as $\bar{x} = f + \bar{\alpha}\hat{r} + \sum_{j=1}^{n} \bar{\alpha}_j r^j$ such that $0 \leqslant \bar{\alpha}_j$, $j = 1, \dots, n$, and $\bar{\alpha}_h = 0$. Let $R(\bar{x}) = \{\sum_{j=1}^{n} \alpha_j r^j \mid 0 \leqslant \alpha_j \leqslant \bar{\alpha}_j,\ j = 1, \dots, n\} + \langle \hat{r} \rangle$. Let us denote by $\mathcal{J}$ the set of all points $\bar{x}$ in S such that $\bar{x}$ is contained in the relative interior of some facet of B. Let $R = \bigcup_{\bar{x} \in \mathcal{J}} R(\bar{x})$.

Lemma 13. *For every $d \in R$, $\pi_l(d) = \psi_B(d)$.*

Proof. We need only to show that, given $\bar{x} \in \mathcal{J}$ and $d \in R(\bar{x})$, $\pi_l(d) = \psi_B(d)$. By symmetry we may assume that $\bar{x}$ is in the relative interior of the facet defined by $a_1(x - f) \leqslant 1$. Let $\bar{\alpha} \in \mathbb{R}$ and $\bar{\alpha}_2, \dots, \bar{\alpha}_n$ nonnegative such that $\bar{x} = f + \bar{\alpha}\hat{r} + \sum_{j=2}^{n} \bar{\alpha}_j r^j$. Because $d \in R(\bar{x})$, there exist $\alpha \in \mathbb{R}$ and $\alpha_1, \dots, \alpha_n$ such that $d = \alpha\hat{r} + \sum_{j=2}^{n} \alpha_j r^j$ and $0 \leqslant \alpha_j \leqslant \bar{\alpha}_j$, $j = 2, \dots, n$.

Note that for $i = 2, \ldots, t$, $(a_1 - a_i)d = \alpha(a_1 - a_i)\hat{r} + \sum_{j=2}^{n} \alpha_j (a_1 - a_i) r^j \geqslant 0$, because $(a_1 - a_i)\hat{r} = 0$ and $(a_1 - a_i)r^j \geqslant 0$. Thus $\psi_B(d) = a_1 d$.

We will show that for every $\lambda < \psi_B(d)$, the set $B(\lambda)$ defined in (7) contains the point $\binom{\bar{x}}{1}$ in its interior. By Theorem 4, this will imply $\pi_l(d) \geqslant \psi_B(d)$, and thus $\pi_l(d) = \psi_B(d)$.

Let $\lambda = \psi_B(d) - \delta$ for some $\delta > 0$. Then $B(\lambda)$ is the set of $x \in \mathbb{R}^{n+1}$ satisfying

$$a_1(x - f) - \delta x_{n+1} \leqslant 1,$$

$$a_i(x - f) + (a_1 - a_i) d x_{n+1} - \delta x_{n+1} \leqslant 1 \quad i = 2, \ldots, t.$$

Substituting $\binom{\bar{x}}{1}$ in the first inequality, we obtain $a_1(\bar{x} - f) - \delta = 1 - \delta < 1$. Substituting in the ith inequality, $i = 2, \ldots, n+1$, we obtain

$$\begin{aligned} & a_i(\bar{x} - f) + (a_1 - a_i)d - \delta \\ &= \bar{\alpha} a_i \hat{r} + \sum_{j=2}^{n} \bar{\alpha}_j a_i r^j + \alpha(a_1 - a_i)\hat{r} + \sum_{j=2}^{n} \alpha_j (a_1 - a_i) r^j - \delta \\ &= \bar{\alpha} a_1 \hat{r} + \sum_{j=2}^{n} \bar{\alpha}_j a_1 r^j - \bar{\alpha}_i (a_1 - a_i) r^i + \alpha_i (a_1 - a_i) r^i - \delta \\ &= a_1(\bar{x} - f) - (\bar{\alpha}_i - \alpha_i)(a_1 - a_i) r^i - \delta \\ &\leqslant 1 - \delta < 1, \end{aligned}$$

where the equality in the third line follows from $a_i \hat{r} = a_1 \hat{r}$ and $a_1 r^j = a_i r^j$ for all $2 \leqslant j \leqslant n$ such that $i \neq j$, while the first inequality on the last line follows from $\alpha_i \leqslant \bar{\alpha}_i$ and $a_i r^i < 1 = a_1 r^i$. $\square$

Note that P is an $n-1$-dimensional simplex in $\mathbb{R}^{n-1} \times \{0\}$ and P does not contain any point of $\mathbb{Z}^{n-1} \times \{0\}$ in its interior. Suppose that P is maximal lattice-free in $\mathbb{R}^{n-1} \times \{0\}$. In this case we can apply the results of §3.2 to identify cases where π_l is a lifting of ψ_B.

Let $\bar{f}$ be the intersection of the line $f + \langle \hat{r} \rangle$ with $\mathbb{R}^{n-1} \times \{0\}$, and let $\bar{r}^j = v^j - \bar{f}$. For every point $\bar{x} \in \mathbb{Z}^{n-1} \times \{0\}$ in the relative interior of one of the facets of P, say the facet defined by $a_h(x - f) \leqslant 1$, $\bar{x}$ can be uniquely written as $\bar{x} = \bar{f} + \sum_{j=1}^{n} \bar{\alpha}_j \bar{r}^j$ such that $0 \leqslant \bar{\alpha}_j$, $j = 1, \ldots, n$, and $\bar{\alpha}_h = 0$. Let $\bar{R}(\bar{x}) = \{\sum_{j=1}^{n} \alpha_j \bar{r}^j \mid 0 \leqslant \alpha_j \leqslant \bar{\alpha}_j, \ j = 1, \ldots, n\}$. Note that $\bar{R}(\bar{x}) = R(\bar{x}) \cap (\mathbb{R}^{n-1} \times \{0\})$. Let $\bar{\mathcal{I}}$ be the set of all points $\bar{x} \in \mathbb{Z}^{n-1} \times \{0\}$ in the relative interior of some of the facets of P. We define $\bar{R} = \bigcup_{\bar{x} \in \bar{\mathcal{I}}} \bar{R}(\bar{x})$. Then $R \supseteq \bar{R} + \langle \hat{r} \rangle$. Hence, if for every $r \in \mathbb{R}^{n-1} \times \{0\}$ there exists $w \in \mathbb{Z}^{n-1} \times \{0\}$ such that $r + w \in \bar{R}$, it also holds that for every $r \in \mathbb{R}^n$ there exists $w^r \in \mathbb{Z}^{n-1} \times \{0\}$ such that $r + w^r \in R$.

Because $\mathbb{R}^{n-1} \times \{0\}$ is the lineality space of $\operatorname{conv}(S)$, Theorem 7 implies that π_l is the unique minimal lifting of ψ_B, and $\pi_l(r) = \psi(r + w^r)$.

The above property holds, for example, when $n = 2$ and P is an interval of length one (as seen in §3.1), when $n = 3$ and P is a maximal lattice-free triangle containing at least four points in $\mathbb{Z}^2 \times \{0\}$, or for general n when P is a unimodular transformation of $\operatorname{conv}(0, (n-1) \cdot e_1, \ldots, (n-1)e_{n-1})$.

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