A Cubical Model for Homotopy Type Theory
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1 Web Page

I encountered difficulty in getting a web page set up and so do not yet have one.

2 Description

I will be working with Prof. Steve Awodey in an effort to construct an internal model of homotopy type theory where types are given a cubical structure (More on this in the following section). While this has little impact outside of the fields of type theory and foundations of mathematics, the fact that homotopy type theory is being used as a system for formalizing proofs – that is, as a basic system whose properties are assumed, much like set theory is commonly used currently – means that having a variety of models of homotopy type theory is useful, if only to other theoreticians. Most notably, a common way to show that some theorem cannot be proven in a particular theory is to show that there is some model of that theory in which the theorem does not hold. While we do not know as yet what such a theorem may be for this particular model, or even if there is one of interest, the techniques used in creating this model may be used elsewhere in order to construct models specifically for this purpose.

3 Definitions and Details

Homotopy type theory is loosely defined as any one of a variety of type theories in which types are treated as having structure similar to the homotopy structure that one can place on a topological space. In some sense, then, we can say that in homotopy type theory, types are interpreted as topological spaces, although it might be more accurate to say that types are interpreted as abstract objects which behave like topological spaces from the perspective of homotopy theory.

To construct a model of homotopy type theory, then, is to define a mapping taking types to some sort of object that we would like to interpret types as, such that that mapping has particular properties – in particular, we would like it to be functorial, and given that, we would like it to be full and faithful. Finally, for this mapping to give an internal model, the objects that we interpret the types as will have to be, in some sense, built of types themselves.

For a mapping to be functorial, we mean that it acts not just on objects (the types) but also on the functions between those objects (in this particular case, closed terms with function type), and that it preserves these functions, in the sense that if $f : a \to b$, and F is functorial, then $F(f) : F(a) \to F(b)$. We also require that the mapping preserve composition, so that $F(g \circ f) = F(g) \circ F(f)$, and that it preserve identity functions $F(1_a) = 1_{F(a)}$.

For a functor to be full, we mean that it has a local surjectivity property – specifically, if $F : A \to B$, then for each $b_1, b_2 \in B$, and for each $g : b_1 \to b_2$, there are $a_1, a_2, f : a_1 \to a_2$ such that $F(f) = g$.

Similarly, for a functor to be faithful, we mean that it has a local injectivity property, in that if $F : A \to B$, then for each $a_1, a_2 \in A$, and each $f_1, f_2 : a_1 \to a_2$, if $F(f_1) = F(f_2)$, then $f_1 = f_2$.

Given these properties – a functor that is full and faithful, we have an embedding. Fullness guarantees that all of the structure of the target comes from somewhere in the source, and faithfulness guarantees that all of the structure of the source is realized in the target. (This is not strictly true, but in a sense, this is all of the "important" structure. Most notably, if $F(A) = F(B)$, then necessarily, $F(1_A) = F(1_B)$, but $1_A \neq 1_B$. However, the fact that F identifies A with B makes this structure irrelevant from the perspective of F.

Finally, a cubical set is a functor from the cube category into the category of sets. Similarly, a cubical type is a functor from the cube category into the category of types. The cube category, while difficult to explain in a simple way, can be thought of as containing all information about what it means to be an n-dimensional cube – how an n-dimensional cube relates to its faces, for example. A cubical set or type, then, is a set or type that behaves like a cube (in a weak sense).

We can extend this idea slightly to give the concepts of symmetric cubes. A symmetric cube is one which, in addition to simply containing the information about how a cube's faces relate to each other and to the larger cube, contains information about how to permute the vertices of the cube. This is less intuitive when thinking about a physical cube, but is useful when working with identity types, as if $w = x = y = z$, there is no reason (when these are treated as the four corners of a square) that they need be in a particular position, and so we may wish to let them be rotated freely.

Similarly, we do some work with globular sets, which are identical to cubical types, but deal with globes (n-spheres) rather than cubes.

4 Goals

I hope to construct an internal model of homotopy type theory in which types are interpreted as cubical types, and will consider the project successful if I am able to do this.

If progress is slower than expected, I hope to at least succeed in considering several approaches to this construction and have created functors from the category of types into the category of cubical types. I will also work to prove or disprove for each of these functors (Which are potential models) their fullness and faithfulness.

If progress is faster than expected, I will work towards finding such a model with additional structure in addition to the cubical structure. In particular, symmetric cubical structure would be an interesting goal to work towards.

5 Milestones

5.1 First Milestone

By the end of this semester, I plan to have constructed a candidate mapping from types into cubical types, although not necessarily to have proven its functoriality.

5.2 January 25th

I plan to have proven the functoriality of the mapping described above.

5.3 February 8th

I plan to have proven the faithfulness of the functor given above.

5.4 February 22nd

I plan to be partway through a proof of fullness of the functor above. Ideally, I will have a proof sketch and be working on formalization. Failing this, I expect to at least have an intuition for how the proof should work.

5.5 March 14th

I plan to have proven the fullness of the functor given above.

5.6 March 28th

I plan to begin writing up a paper for the result above.

5.7 April 11th

I plan to have the paper finished and under revision.

5.8 April 25th

I plan to have a finalized paper at this point, and to begin working on further results.

6 Literature Search

This project primarily builds on the work of Benno van den Berg, Richard Garner, and Nicola Gambino. In particular, the paper "Types are weak omega-groupoids"[2] deals with a similar idea, where types are interpreted as omega-groupoids. This is similar to my project, except that while I am working to give types a cubical structure, to interpret types as omega-groupoids requires only globular structure. The other notable paper, "The identity type weak factorization system"[1], is useful in that it defines from the category of contexts of a type theory what an identity context is, analogous to an identity type. This will be useful as the category of contexts is in some ways better-behaved than the category of types to which it is associated – most notably, some equalities which are only propositional (that is, they can be proven) in the category of types are definitional (that is, the proof is that they hold by definition) in the category of contexts.

This is likely not all of the reading that I will need to do, but it is likely that other reading will come up unexpectedly, to solve a particular difficulty in this project, rather than being background material.

7 Resources Needed

As this is a purely theoretical research project, no resources are needed.

References

[1] Nicola Gambino and Richard Garner, *The identity type weak factorisation system*, Theoret. Comput. Sci. **409** (2008), no. 1, 94–109. MR 2469279 (2011d:03011)

[2] B. van den Berg and R. Garner, *Types are weak omega-groupoids*, ArXiv e-prints (2008).