Equivalent Certain Values and Dynamic Irreversibility

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Dynamic Irreversibility

- Many dynamic decisions involve:
  - Irreversibility
  - Uncertainty
- Examples:
  - R&D investments
  - Capacity allocation
  - Long auctions
- These problems are often difficult to analyze
- Need for a tractable methodology

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- Characterization of properties and comparative statics of ECV
    - ECV goes down if uncertainty goes up
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- Characterization of properties and comparative statics of ECV
  - ECV goes down if uncertainty goes up
  - ECV goes up as we get closer to the deadline
- Show irreversibility is analogous to information loss
  - Act as if you have worse information
- Application to dynamic auction design

Outline

- **Model**
- Key Results
- Application to Dynamic Auctions

General Model Structure

- Time: Continuous $[0, T]$
- Decision times: Random $\tau_0, \tau_1, ...$
- Actions: $a_\tau \in A$
  - A: totally ordered set
  - $a_{\tau'} \geq a_\tau$ for $\tau' > \tau$ (irreversible actions)
- a_T: the final action
- Final payoff: $U(v_T, a_T)$

Assumptions on Payoff Function

- $U(v, a)$:
  - Linear in v
  - Supermodular in v and a
  - Admits a maximum with respect to a for all v.

Information Arrival and Decision Times

- Joint stochastic processes on $[0, T]$
- Decision times: Stopping times $\{\tau_n(\omega)\}$, where $\tau_{n+1}(\omega) > \tau_n(\omega)$
  - modeled as jumps of counting process $\{\eta(t, \omega)\}$

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- Filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ represents available information
  - increasing σ-algebras on Ω with the property that $\mathcal{F}_t \subset \mathcal{F}_{t+s} \subset \mathcal{F}$.
  - $\{\mathcal{F}_t\}$ generated by $\{\eta(t, \omega), \tilde{v}(t, \omega)\}$

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Assumption: $\{v_n, \tau_n\}$ follows a joint Markov process, i.e.,

$$P\left(v_{n+1} = v', \tau_{n+1} = \tau' | \mathcal{F}_{\tau_n}\right) = P\left(v_{n+1} = v', \tau_{n+1} = \tau' | v_n, \tau_n\right).$$

$\Rightarrow$ can identify decision nodes with pairs (v_n, τ_n) corresponding to the realized signal and time in the last arrival.

Information Arrival and Decision Times

- Decision times: exogenous
- However, the specification is still flexible
  - Allows correlation between decision times and expected values
  - Captures varying eagerness to revise strategy based on value
  - Allows for nonstationary Markov process (more arrival rate closer to deadline)

Decision Strategies and Optimal Choice

- Decision strategy s:
  - Specifies action $s(v_n, \tau_n)$ at each decision node
- Prevailing action at time t: $a(s,t) = \max\{s(v_n, \tau_n) | \tau_n \leq t\}$
- Final choice: $a(s,T)$
- $\boldsymbol{S}$: Set of strategies satisfying these conditions
- for each realized path ω:
  - value $U\left(v\left(T,\omega\right), a\left(s,T,\omega\right)\right)$,
  - where $a\left(s,T,\omega\right) = \sup\left\{s\left(v_n\left(\omega\right), \tau_n\left(\omega\right)\right) | \tau_n \leq T\right\}$.

Optimal Decision Strategy:

$$\sup_{s \in \boldsymbol{S}} E_0 U(v(T), a(s,T))$$

Examples of Dynamic Problems

- Entry Decisions and Search
  - Random entry opportunities or search offers
  - Binary action space: $A = \{0, 1\}$
- Bidding in Long Auctions
  - Changing bidder values over time
  - Increasing bids only
- Irreversible Investment
  - Random investment opportunities
- General Contest and Teamwork
  - Effort exertion at random times, uncertain prize value or alternatives
- Sequential Trading Commitments
  - Random opportunities to sell future deliveries

Key Features: Random decision times, irreversibility, changing values

Methodology

- Two step procedure:
  1. Identify "equivalent certain values" (ECV)

  2. Find corresponding optimal choices for ECV

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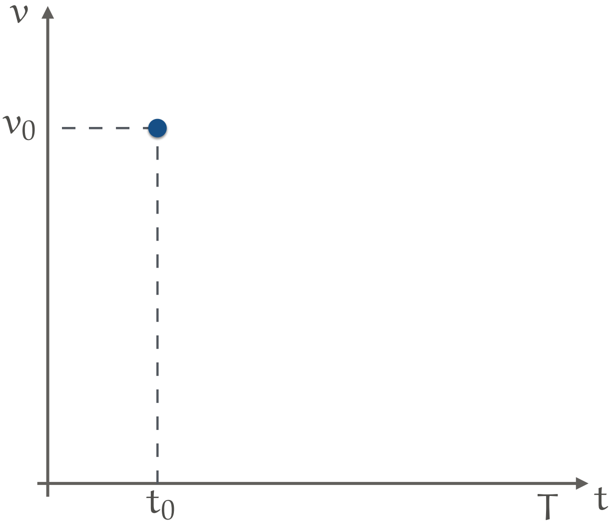
- Two step procedure:
  1. Identify "equivalent certain values" (ECV)
     - ECVs partition the state space into indifference classes
     - Agent's optimal action is the same for any point with the same ECV
     - We define it implicitly by a recursive problem (dynamic)
  2. Find corresponding optimal choices for ECV
     - Optimal action if no future opportunities and your value is ECV
     - This is a static problem
     - The particular payoff function, U, is only used in this step

Illustrating ECVs



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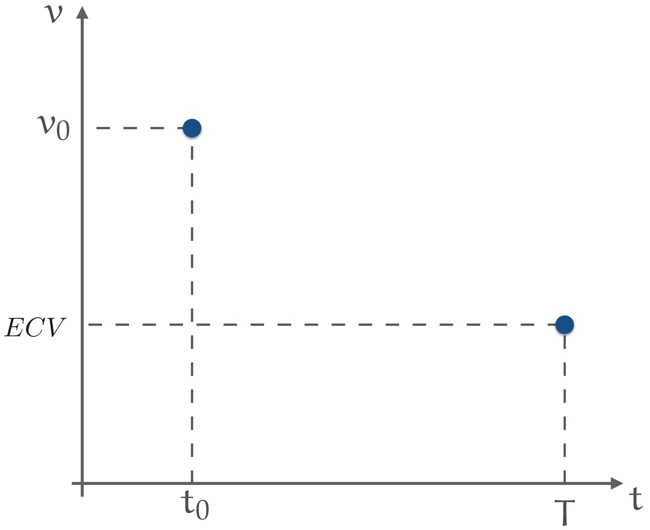

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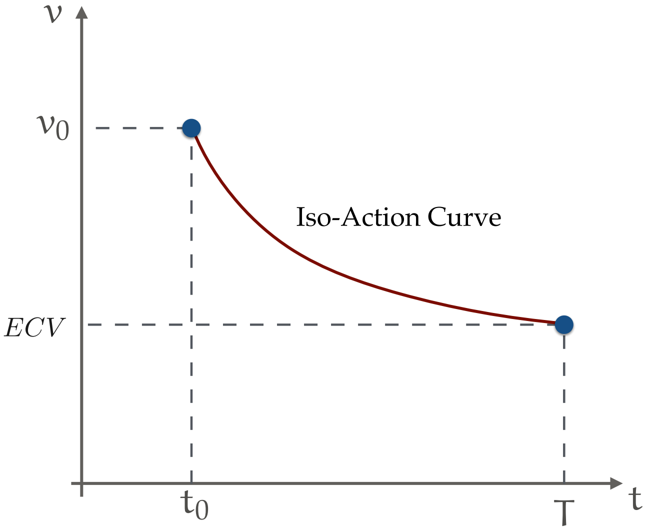

Defining ECV

- Define ECV function: $e(v, t)$
- Self-generated expectation property:

$$e(v, t) = E(v_T | \omega \in H(e(v, t), v, t))$$

- Where $H(e, v, t)$ is the set of ωs where all future arrivals (if any) have lower ECV

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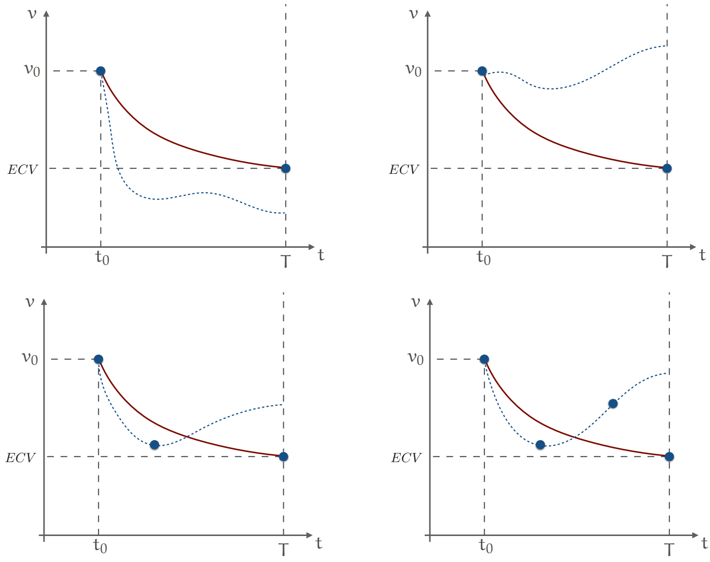
    - This is the set in which the action taken at t is going to be the final action

Self-Generated Property of ECV



$H(e, v, t)$: set of ωs where all future arrivals (if any) have lower ECV

Calculating ECVs

- Auxiliary functional equation given $W(\varepsilon, v, t)$:

$$\begin{aligned} W(\varepsilon, v, t) &= \int_t^T \min\left(W(\varepsilon, v', \tau'), 0\right) dP(v', \tau' | v, t) \\ &\quad + \int_{N(v,t)} \left(v_T(\omega) - \varepsilon\right) d\Pi(\omega | v, t) \end{aligned}$$

- where $N(v, t)$ denote the set of paths $\omega \in \Omega$ such that there are no arrivals after (v, t)
- $e(v, t)$ defined implicitly by $W(e(v, t), v, t) = 0$

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- where $N(v, t)$ denote the set of paths $\omega \in \Omega$ such that there are no arrivals after (v, t)
- $e(v, t)$ defined implicitly by $W(e(v, t), v, t) = 0$
- Proposition 1: Unique solution to the above functional equation exists and $W(e(v, t), v, t) = 0$ satisfies self-generated expectation property. Assumption

Optimal Solution

- Static problem:

$$\tilde{S}(v) = argmax_a U(v, a)$$

- Optimal dynamic strategy: $S(v,t) = \tilde{S}(e(v,t))$
- **Theorem 1**: This strategy is optimal for any payoff function (given the assumptions mentioned earlier)

Proof Sketch for Theorem 1

1. Consider a decision node (v, t) and alternative action $a_2 \neq a_1 = S(v, t)$
2. Show that one-period deviation is not an improvement
3. Use properties of ECVs and supermodularity of payoff function

Key Implications

- ECVs partition the state space into indifference classes
- Optimal strategy depends only on ECVs, not full history
- Reduces dynamic problem to a sequence of static problems
- Allows for tractable analysis of a wide range of dynamic problems

Embedding in Games

Game: $\Gamma = \big(I, \{A_i\}_{i\in I}, \{Z_i\}_{i\in I}, \{P_i\}_{i\in I}, \{u_{iT}\}_{i\in I}\big)$

- I set of players
- A_i action spaces
- Z_i space of values
- P_i transition process on $Z_i \times [0, T]$
- $u_{iT}(a_i, a_{-i})$ final payoff functions

Equilibrium

- Strategies $S_i : Z_i \times [0, T] \to A_i$
- Expected payoffs $u_i(S_i, S_{-i}) = \mathbb{E}_0 u_{iT}(v_{iT}, a_{iT}, a_{-iT} | S_i, S_{-i})$
- Nash equilibrium in normal form game
  - $u_i(S_i, S_{-i}) \geq u_i(S_i', S_{-i})$ for all $S_i' \in \mathbf{S_i}$.
- High dimensional problem

Mapping into Bayesian Game

Equivalent values: For every history ω,

$$v_i(\omega) = \max\{e_i(v_n(\omega), \tau_n(\omega))\}$$

This induces distributions of values Ψ_i for each player.

Bayesian Game: $\Gamma_B = \left(I, \{\Psi_i\}_{i\in I}, \{A_i\}_{i\in I}, \{u_{iT}\}_{i\in I}\right)$
Assumption: $u_{iT}(v_i, a_i, a_{-i})$ are linear in v_i and supermodular in (v_i, a_i).
Theorem: Given equilibrium strategies $\left\{\tilde{S}_i\right\}_{i\in N}$ of Γ_B the strategies defined by $S_i(v,t) = \tilde{S}_i(e_i(v,t))$ are an equilibrium for Γ

Decomposition

- Our result decomposes the problem of finding an equilibrium to Γ into two steps:
    1. A dynamic decision problem to find the equivalent final values $e(v, t)$
    2. A static equilibrium determination of the Bayesian game
- Result holds without privately observed actions when Bayesian Game has an equilibrium in weakly dominating strategies (e.g. second-price auction)
- Also for Anonymous Sequential games

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- Strict under fairly general regularity conditions.
- Determinants of shading:
  - Variance of innovations – precision of signals
  - Arrival process for action times
- ECV goes up over time for a given expected value

Irreversibility and Information Loss

- **Proposition**: Distribution of final values $\{\bar{v}(\omega)\}$ is mean-preserving spread of $\{\bar{e}(\omega)\}$
  - $\bar{v}(\omega)$: final value associated with any path
  - $\bar{e}(\omega)$: final ECV associated with any path
- Irreversibility constrains actions, limiting use of information
- Agent acts as if they had worse information than with reversible actions

Effect of Increasing Arrival Rates

- **Proposition**: More frequent arrivals result in:
  - Higher shading initially
  - More frequent actions
  - Mean-preserving spread of final actions

Outline

- Model
- Key Results
- **Application to Dynamic Auction Design**

Dynamic Second-Price Auctions

- N bidders with independent private values
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- Standing bid: maximum of ECVs among decision nodes
- Results in:
  - Revenue equivalence holds under standard conditions
  - Optimal reserve price similar to static case, using $\bar{e}(\omega)$ distribution

Design Implications

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  - Mean-preserving spread of bids
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- Increasing arrival rates:
  - Also leads to mean-preserving spread of bids
  - Effects depend on number of bidders
  - Many bidders: higher arrival rates leads to higher winning bids

Literature Review

- Dynamic decision problems with irreversibility:
  - Arrow and Fisher (1974), Henry (1974): Option value in irreversible decisions
  - Dixit et al. (1994): Investment under uncertainty
- Revenue management and dynamic pricing:
  - Elmaghraby and Keskinocak (2003), Den Boer (2015): Surveys
  - Zhao and Zheng (2000): Dynamic pricing with limited capacity
- Random Opportunities: Ockenfels and Roth (2006), Ambrus, Ishii and Burns (2014), Groeger and Miller (2015), Revision games: Kamada and Kandori (2020), Kapor and Moroni (2016)
- Dynamic across auctions:
  - Jofre-Bonet and Pesendorfer (2003), Zeithammer (2006), Said (2011), Hendricks and Sorensen (2018), Coey, Larsen and Platt (2020), Backus and Lewis (2024)

Final Remarks

- Analyzed a class of dynamic problems with irreversible actions
- Embedding in games with privately observed actions
- Decomposition: dynamics/equilibrium
- Can relax assumption of privately observed actions
  - When Bayesian Game has an equilibrium in weakly dominating strategies
  - Anonymous Sequential/Mean field games
  - Oblivious equilibrium?
- Design applications
  - Optimal dynamic auction

Thank You!

Assumption

The following properties hold:

1. There exists $\delta > 0$ such that $\Pi\left(N\left(v,t\right)|v,t\right) > \delta$ for all (v,t),
2. The integral $\int_{N(v,t)} \left(v_T\left(\omega\right)\right) d\Pi\left(\omega|v,t\right)$ is continuous in v, t, and
3. The Markov process, $P\left(v', t'|v,t\right)$, is continuous in the topology of weak convergence.