21-237: Math Studies Algebra I August 29, 2022

Lecture 1 : Introduction to Groups

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1 Foundations

1.1 Definitions

Given a set X, a **binary operation** on X is a function from X^2 to X.

Let $*$ be a binary operation on X[1]. $e \in X$ is a **two-sided identity** element if $ex = xe = x$ for all $x \in X$. Note: a two-sided identity might exist (e.g. 1 for multiplication over the reals) and might not (e.g. addition over the positive reals).

Let e be the two-sided identity of $(X, *)$. Let $x \in X$. y is a **two-sided inverse** of x iff $xy = yx = e$.

A **group** is a set G equipped with a binary operation $*$ such that (1) $*$ is associative, (2) $*$ has an identity, (3) every $g \in G$ has an inverse. The identity is called 1_G. The inverse of g is called g^{-1}.

1.2 Theorems

For any binary operation, there is at most one two-sided identity.

Let e, f be two-sided identities. Then $e = ef = f$.

If y_1, y_2 are both inverses for x and $*$ is associative, then $y_1 = y_2$.

$y_1 = y_1 e = y_1(xy_2) = (y_1 x)y_2 = ey_2 = y_2$.

1.3 Examples of groups

Note: The group axioms can and should be verified for the following examples.

1. **Trivial group**: $G = \{1\}$, $1 * 1 = 1$. The trivial group is sometimes called 1.
2. $(\mathbb{Z}, +)$
3. (Meta-example) For any mathematical structure S, the symmetries of S form a group.
   (a) Let X be any set. A permutation of X is a bijection from X to X.
   Let $\Sigma_X = \{\pi : \pi \text{ is a permutation of } X\}$. $(\Sigma_X, \circ)$ is a group.
   (b) A metric space is a set X equipped with $d : X^2 \to \mathbb{R}$ such that $d(x, y) \geq 0$ for all $x, y \in X$, $d(x, y) = 0 \iff x = y$, $d(x, y) = d(y, x)$ for all $x, y \in X$, and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
   Let (X, d) be a metric space. An isometry of (X, d) is $\pi \in \Sigma_X$ such that $d(\pi(x), \pi(y)) = d(x, y)$ for all $x, y \in X$. Isometries of (X, d) form a group under composition. In particular, they form a subgroup of $(\Sigma_X, \circ)$.
   (c) Let $(G, *)$ be a group. $\pi \in \Sigma_G$ is an automorphism of $(G, *)$ if and only if $\pi(gh) = \pi(g)\pi(h)$ for all $g, h \in G$.
   $\text{Aut}(G)$ is a group of automorphisms of G with composition as its operator.

[1] We use infix notation for application, i.e. we write $*(x, y)$ as $x * y$ or xy.