21-237: Math Studies Algebra I September 21, 2022

Lecture 10 : Ideas that have merit in their own right

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1 Conjugation action on subgroups

Let G be a group. Let G act on the set $\{H : H \leq G\}$ via $g \cdot H = H^g$. Verify axioms as an exercise.

Note: $H^g \leq G$, and H is isomorphic to H^g ($H \simeq H^g$) via isomorphism $h \mapsto h^g$ (see that $h' \mapsto h'^{g^{-1}}$ is a two-sided inverse).

If $H \leq G$, the stabilizer of H (for this action) is $\{g : H^g = H\} = N_G(H)$, the **normalizer** of H.

Easy facts: $N_G(H) \leq G$, $H \triangleleft N_G(H)$, $N_G(H)$ is the largest subgroup of G in which H is normal.

Theorem: Let $H, K \leq G$ and $K \leq N_G(H)$, then $HK = KH \leq G$.

Proof: We have that $K \leq N_G(H)$ and $H \triangleleft N_G(H)$. By homework, $KH = HK \leq N_G(H) \leq G$.

Recall: If $N \triangleleft G$, we formed the **quotient group** G/N and **quotient homomorphism** $\phi_N : G \to G/N$ via $\phi_N(g) = gN$.

Theorem: Let $N \triangleleft G$. Then, there is a bijection between the subgroups of G/N and the subgroups of G which contain N. Normal subgroups of G/N correspond to normal subgroups of G which contain N.

Let $N \leq M \leq G$. Easily, $N \triangleleft M$ and $M/N \leq G/N$.

As $N \leq M$, M is the union of cosets of N in M. So M/N is the set of cosets of N in G which are subsets of M. In particular, if $N \leq M_1 \leq G$ and $N \leq M_2 \leq G$ for $M_1 \neq M_2$, then $M_1/N \neq M_2/N$.

Let $\overline{M} \leq G/N$. $\overline{M}$ is a set of cosets of N in G. Let $M = \bigcup \overline{M} = \{m \in G : \exists C \in \overline{M}. m \in C\}$.

We claim that $N \leq M \leq G$: As $\overline{M} \leq G/N$, $1_{G/N} = N \in \overline{M}$, so $N \leq M$. Let $m_1, m_2 \in M$. Then, $\exists C_1, C_2 \in \overline{M}. m_1 \in C_1, m_2 \in C_2$, i.e. $m_1N, m_2N \in \overline{M}$. Note that $(m_1N)(m_2N) = m_1m_2N \in \overline{M}$, so $m_1m_2 \in M$. Inverses are similar.

It is easy to show $\overline{M} = M/N$.

So far, we have an explicit bijection between $\{\overline{M} : \overline{M} \leq G/N\}$ and $\{M : N \leq M \leq G\}$.

Verify: When $\overline{M} = M/N$ for $N \leq M \leq G$, $\overline{M} \triangleleft G/N \iff M \triangleleft G$.

2 Players in the proof of Sylow's Theorem

Let $|G| < \infty$, p prime, $p \mid |G|$ and p^t is the highest power of p dividing $|G|$.

Let $\Sigma = \{P : P \leq G, |P| \text{ is a positive power of } p\}$. Note that Cauchy's theorem gives $p \in G$, where $|p| = G$, so $\langle p \rangle \in \Sigma$, so $\Sigma \neq \emptyset$. Σ forms a poset under inclusion.

Let $\Omega = \{Q \in \Sigma : Q \text{ is maximal under inclusion}\}$. Q is maximal in $(\Sigma, \subseteq)$ when for all $P \in \Sigma$, $Q \subseteq P \implies Q = P$. For all $P \in \Sigma$, there is $Q \in \Omega$, $P \leq Q$.