Lecture 12 : Applications of Sylow's Theorem

Lecturer: James Cummings
Scribe: Rajeev Godse

1 Finishing the proof of Sylow's Theorem

Recall: for finite G, where prime p divides $|G|$ and p^t is the highest power of p dividing G, $\Omega = \{H \leq G : H \text{ is a maximal } p\text{-subgroup}\}$, $\text{Syl}_p(G) = \{H \leq G : |H| = p^t\}$.

For $H \in \Omega$, $|\Omega| = [G : N_G(H)]$ by orbit-stabilizer, and $|\Omega| \equiv 1 \bmod p$.

Claim 9: $\Omega = \text{Syl}_p(G)$. *Proof*: Clearly, $\text{Syl}_p \subseteq \Omega$, just by LaGrange's theorem. To show $\Omega \subseteq \text{Syl}_p$, let $H \in |\Omega|$ and AFSOC $H \notin \text{Syl}_p$. By definition, the largest power of p dividing $|H|$ is p^s for $0 < s < t$. Note also that $H \triangleleft N_G(H)$.

$|\Omega| = [G : N_G(H)] \equiv 1 \bmod p$, so p^t divides $|N_G(H)|$. Applying Cauchy's theorem to $N_G(H)/H$ (which has order divisible by p), $N_G(H)/H$ has a subgroup of order p. Using the bijection between subgroups of $N_G(H)/H$ and subgroups of $N_G(H)$ that contain H, there is K such that $H \leq K \leq N_G(H)$ and $|K/H| = p$. Now $K = p \cdot p^s = p^{s+1}$, so $K \in \Sigma$ and $K \nsubseteq H$ but $K \neq H$. This yields a contradiction, so $\Omega \subseteq \text{Syl}_p$, and thus $\Omega = \text{Syl}_p(G)$.

2 Corollaries

Corollary 1

Let G be a finite group where prime p divides $|G|$. Let $H \in \text{Syl}_p(G)$. if $H \triangleleft G$, then $\text{Syl}_p = \{H\}$.

Proof: All Sylow p-subgroups are conjugate.

Corollary 2

If $\text{Syl}_p(G) = \{H\}$, then H is characteristic in G.

Proof: Automorphisms respect subgroups and order, so in general, they permute the Sylow p-subgroups. If there is only one, it must be mapped to itself by all automorphisms.

Corollary 3

If $H \in \text{Syl}_p(G)$, then $\text{Syl}_p(N_G(H)) = \{H\}$. In particular, $H \leftrightarrow N_G(H)$ sets up a bijection between Sylow p-subgroups and normalizers of Sylow p-subgroups.

3 Structure of groups with prime power orders

Recall: If $|G| = p^t$ for p prime, $t > 0$, sometimes denoted "G is a non-trivial finite p-group", then $Z(G) \neq 1$.

Theorem: If G is a non-trivial finite p-group, there is $N \triangleleft G$, such that $|N| = p$.

Proof: Apply Cauchy to $Z(G)$ and find $N \leq Z(G)$ with $|N| = p$. $n^g = n$ for all $n \in N$, so $N \triangleleft G$.

Theorem: Let $|G| = p^s$ for p prime. Then there exist subgroups $(N_i)_{0 \leq i \leq s}$ such that $N_i \triangleleft G$, $|N_i| = p^i$, and $N_i \leq N_{i+1}$ for $0 \leq i \leq s$. ($N_0 = 1$, $N_s = G$)

Proof: By induction on s. If $s = 0$, take $N_0 = 1$.

For $s > 0$, there is $N \triangleleft G$, $|N| = p$ by Cauchy's theorem. Apply IH to G/N, since $|G/N| = p^{s-1}$, yielding a chain of normal subgroups of G/N which correspond to normal subgroups of G containing N. So the chain of normal subgroups of G/N correspond to normal subgroups of G containing N that all contain each other and have powers ranging from p to p^s. Then take $N_0 = 1$ and we have another chain.