21-237: Math Studies Algebra I September 28, 2022

Lecture 13 : Commutators and abelian normal subgroups

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1 Commutators

Recall: For $a, b \in G$, the commutator of a and b is $[a, b] = aba^{-1}b^{-1}$.

The **commutator subgroup** of G (sometimes called the **derived subgroup**), denoted $[G, G]$, is the subgroup generated by $\{[a, b] : a, b \in G\}$. Note that not all elements of $[G, G]$ are of the form $[a, b]$.

Fact: $[G, G]$ is a characteristic subgroup of G. *Proof sketch*: If $\alpha \in \mathrm{Aut}(G)$, $\alpha([ab]) = [\alpha(a), \alpha(b)]$.

Fact: For $N \triangleleft G$, G/N is abelian iff $[G, G] \leq N$.

Proof: G/N is abelian iff $[aN, bN] = N$ iff $[a, b] \in N$ for all $a, b \in G$ iff $[G, G] \leq N$.

A **subnormal series** in G is a finite sequence of subgroups $G_0, G_1, \ldots, G_t$ such that $G_0 = 1$, $G_t = G$, and $G_i \triangleleft G_{i+1}$ for $0 \leq i < t$.

A **normal series** is a subnormal series as above in which $G_i \triangleleft G$ for $0 \leq i \leq t$.

A **characteristic series** is a sequence of subgroups $G_0, G_1, \ldots, G_t$ such that $G_0 = 1$, $G_t = G$, and G_i char G for $0 \leq i \leq t$.

G is **solvable** if and only if there is a subnormal series $(G_i)_{0 \leq i \leq t}$ such that G_{i+1}/G_i is abelian for $0 \leq i < t$.

Let G be a group. The **derived** series is the sequence of subgroups $(G^{(i)})_{i \in \mathbb{N}}$ given by $G^{(0)} = G$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.

Fact: $G^{(i)}$ char G for all i.

Fact: G is solvable iff there is $i \in \mathbb{N}$ such that $G^{(i)} = 1$.

Proof: Suppose $G^{(i)} = 1$. Consider the series $G^{(i)} = 1, G^{(i-1)}, \ldots, G^{(0)} = G$. $G^{(i)}/G^{(i-1)}$ is abelian just by definition of the commutator.

Conversely, let G be solvable and fix a subnormal series $H_0, \ldots, H_s$ with H_{i+1}/H_i abelian for $0 \leq i < s$. As H_{i+1}/H_i is abelian, $[H_{i+1}, H_{i+1}] \leq H_i$. We show by induction on j that $G^{(j)} \leq H_{s-j}$.

For the base case of $j = 0$, $G^{(0)} \leq H_s = G$. Then suppose $G^{(j)} \subseteq H_{s-j}$ for $j \geq 0$. Then, $G^{(j+1)} = [G^{(j)}, G^{(j)}] \subseteq [H_{s-j}, H_{s-j}] \subseteq H_{s-j-1} = H_{s-(j+1)}$.

Extension problem: Given groups H and N, find a group G such that there is $\overline{N} \triangleleft G$, $\overline{N} \simeq N$, $G/\overline{N} \simeq H$.