21-237: Math Studies Algebra I September 30, 2022

Lecture 14 : A mixed bag

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1 Extension problem

Extension problem: Given groups H and N, find a group G such that there is $\overline{N} \lhd G$, $\overline{N} \simeq N$, $G/\overline{N} \simeq H$.

One solution turns out to be $G = H \times N$, $\overline{N} = 1 \times N$. More general (from HW) is $G = H \ltimes_\phi N$ for any IM $\phi : H \to \text{Aut}(N)$.

2 Simple groups

G is **simple** if $G \neq 1$ and for all $N \lhd G$, $N = 1$ or $N = G$.

Easy fact: Suppose $A \lhd B$ and B/A is simple. Then $B \neq A$, i.e. $B/A \neq 1$, and for all $C \lhd B$ such that $A \lhd C$, $C = A$ or $C = B$ since the normal subgroups of B/A is are in corresondence with the normal subgroups of B containing A.

Easy corollary: If $G_0 \lhd G_1 \lhd \ldots \lhd G_t$ is a subnormal series such that G_{i+1}/G_i is simple for $0 \leq i < t$, then no new groups can be inserted in the series while preserving subnormality.

3 Composition series

Let G be a group. A **composition series** for G is a subnormal series $G_0 = 1 \lhd G_1 \lhd \ldots \lhd G_t = G$ such that G_{i+1}/G_i is simple for all i, $0 \leq i < t$.

Fact: If G is finite, G has a composition series.

Theorem (Jordan-Hilder): If $(G_i)_{0 \leq i \leq s}$ and $(H_j)_{0 \leq j \leq t}$ are composition series for the same group, then a) $s = t$ and b) there is a permutation π of $\{0, \ldots, s-1\}$ such that $G_{i+1}/G_i \simeq H_{\pi(i)+1}/H_{\pi(i)}$.

Important note: The quotients of a composition series do not uniquely determine the group. For example, both S_3 and C_6 have composition series with quotients of C_2 and C_3.

4 Solvability

Recall: G is solvable iff there is a subnormal $G_0 = 1 \lhd \ldots \lhd G_t = G$ with G_{i+1}/G_i abelian for all i.

Theorem:

(a) If G is solvable and $H \leq G$, then H is solvable.

(b) If G is solvable and $N \lhd G$, then G/N is solvable.

(c) If $N \lhd G$ and both N and G/N are solvable, then G is solvable.

Proof: For (a) and (b), let $(G_i)_{0 \leq i \leq t}$ witness the solvability of G. As G_{i+1}/G_i abelian, $[G_{i+1}, G_{i+1}] \leq G_i$.

For (a), let $H_i = G_i \cap H$. Tedious verification shows that $(H_i)_i$ is subnormal and $[H_{i+1}, H_{i+1}] \subseteq [G_{i+1}, G_{i+1}] \cap H \subseteq H_i$, so H_{i+1}/H_i is abelian.

For (b), let $G_i^* = \phi_N[G_i] = \frac{G_i N}{N}$. $[G_{i+1}^*, G_{i+1}^*] = \frac{[G_{i+1}, G_{i+1}]N}{N} \subseteq \frac{G_i N}{N} = G_i^*$.

For (c), as N is solvable, fix $(N_i)_{0\leq i\leq s}$ witnessing N's solvability. Then, $[N_{i+1}, N_{i+1}] \leq N_i$.

As G/N is solvable, use the correspondence between subgroups of G/N and subgroups of G containing N to find $(\overline{G_j})_{0\leq j\leq t}$ where $\overline{G_0} = N$, $\overline{G_t} = G$ such that $\overline{G_i}/n \triangleleft \overline{G_{i+1}}/N$ and $[\overline{G_{i+1}}/N, \overline{G_{i+1}}/N] \leq G_i/N$. We can then verify that $\overline{G_i} \triangleleft \overline{G_{i+1}}$ and $[\overline{G_{i+1}}, \overline{G_{i+1}}] \leq \overline{G_i}$.

Then, $N_0 = 1, \ldots, N_s = N = \overline{G_0}, \ldots, \overline{G_t} = G$ witnesses that G is solvable.

5 Free groups

Now, for something completely different.

Consider $(\mathbb{Z}, +)$. This is cyclic, generated by 1.

Theorem: For any group H, any element $h \in H$, there is a unique HM $\phi : (\mathbb{Z}, +) \to H$ such that $\phi(1) = h$.

Proof: If ϕ exists, easy to see $\phi(n) = h^n$ for all $n \in \mathbb{Z}$. It is similarly easy to easy that this defines the HM we want.