Lecture 15 : Free groups part 1

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1 Freedom

Recall: for any G and any $g \in G$, there is a unique $\phi : \mathbb{Z} \to G$ such that $\phi(1) = g$, namely $\phi = n \mapsto g^n$.

For two elements, say that G is **free** on a two-element set if there exists $a, b \in G$ such that $G = \langle a, b \rangle$ and for any H and any $c, d \in H$, there is unique HM $\phi : G \to H$, $\phi(a) = c, \phi(b) = d$.

Easy: If G free on a, b and G' is free on a', b', then there is a unique IM $\phi : G \simeq G'$ such that $\phi(a) = a'$ and $\phi(b) = b'$. *Proof*: Let ϕ be unique HM from G to G' where $\phi(a) = a'$, $\phi(b) = b'$. Let ψ be unique HM from G' to G where $\psi(a') = a$ and $\psi(b') = b$. $\phi \circ \psi$ and $\psi \circ \phi$ both fix both a and b, so they are both the identity.

2 Remember 251?

Let X be an arbitrary set (think of X as a set of symbols). A **word** on X is a finite sequence (possibly empty) of elements of $X \times \mathbb{Z}$.

Notation: We write $x_1^{n_1} \dots x_t^{n_t}$ for the word $(x_1, n_1) \dots (x_t, n_t)$.

A word $x_1^{n_1} \dots x_t^{n_t}$ is **reduced** ifthere is no i such that $n_i = 0$ and there is no i such that $x_i = x_{i+1}$.

Let w and w' be words. Then, w' is a 1-step **reduction** of of w if w' is obtained from w by deleting a single entry $(x, 0)$ or w' is obtained from w by replacing successive entries (x, i) and (x, j) by $(x, i + j)$. Similarly, w' is a t-step reduction if w' can be obtained from w by a series of t 1-step reductions.

Key point (to be proven soon): For any word w, there is a unique reduced word w' such that w' can be obtained from w by a series of 1-step reductions.

One way to make a free group is to let the underlying set be the set of reduced words and the group operation be "concatenate and reduce."

3 Starting the proof

Let W be the set of reduced words.

For each $x \in X$, we define $\alpha_x, \beta_x \in \Sigma_W$ via $\alpha_x(w) = \begin{cases} w' & w = x^{-1}w' \\ x^{n+1}w' & w = x^n w', n = -1 \\ xw & \text{otherwise} \end{cases}$ and β_x defined in a similar way for prepending x^{-1}.