21-237: Math Studies Algebra I October 5, 2022

Lecture 16 : Free groups part 2

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1 Our main construction

Given set X, we want a group G generated by elements $(g_x : x \in X)$ such that for any group H and any function $f : X \to H$, there is a unique HM $\phi : G \to H$ such that $\phi(g_x) = f(x)$ for all $x \in X$.

Recall: W is the set of reduced words on X. For each $x \in X$, we define $\alpha_x : W \to W$ where $\alpha_x(w)$ is the result of reducing $x^1 w$.

Formally, $\alpha_x(w) = \begin{cases} w' & w = x^{-1}w' \\ x^{n+1}w' & w = x^n w', n \neq -1 \\ xw & \text{otherwise} \end{cases}$.

We can also define $\beta_x : W \to W$, where $\beta_x(w)$ is the result of reducing $x^{-1}w$.

Formally, $\beta_x(w) = \begin{cases} w' & w = x^{1}w' \\ x^{n-1}w' & w = x^n w', n \neq 1 \\ xw & \text{otherwise} \end{cases}$.

An easy analysis shows that $\alpha_x \circ \beta_x = \beta_x \circ \alpha_x = 1$ for all $x \in X$.

Since α_x has a two-sided inverse, $\alpha_x \in \Sigma_W$.

Let $G =$ subgroup of Σ_W generated by $\{\alpha_x : x \in X\}$. We show this G achieves our goals.

2 A series of innocuous claims

Let $v = x_1^{n_1} \dots x_t^{n_t}$ be a word. Then, let $\alpha_v = \alpha_{x_1}^{n_1} \dots \alpha_{x_t}^{n_t}$. These claims are meant to be easy.

Claim 1: $G = \{\alpha_v : v \text{ is a word}\}$. By convention, $\alpha_\epsilon = 1$. *Proof*: Definition of generated subgroup.

Claim 2: For words v, v', $\alpha_v \circ \alpha_{v'} = \alpha_{vv'}$, where vv' is the concatenation of the two words. *Proof*: Basic equational reasoning.

Claim 3: $\alpha_{x^0} = 1$. *Proof*: $\alpha_x^0 = 1$.

Claim 4: $\alpha_{x^n x^m} = \alpha_{x^{n+m}}$. *Proof*: Claim 2.

Claim 5: If v' is obtained from v by 1-step reduction, $\alpha_v = \alpha_{v'}$. *Proof*: Claims 3, 4.

Claim 6: If v' is obtained from v by a finite series of 1-step reductions, $\alpha_v = \alpha_{v'}$. *Proof*: Induct using claim 5.

Claim 7: If v is reduced, then $\alpha_v(\epsilon) = v$. *Proof*: Let $v = x_1^{n_1} \dots x_t^{n_t}$ be reduced. Then, $\alpha_v = \alpha_{x_1}^{n_1} \dots \alpha_{x_t}^{n_t}$. In essence, there is no room for reduction in $\alpha_v(\epsilon)$, since v is reduced (specifically, $x_{t-1} \neq x_t$). Thus, iteratively applying the permutations that compose α_v on ϵ yields $x_1^{n_1} \dots x_t^{n_t}$, which is v.

Claim 8: If v is a word and v', v'' are both reduced words obtained from v by some series of reduction steps, then $v' = v''$. *Proof*: $v' = \alpha_{v'}(0) = \alpha_v(0) = \alpha_{v''}(0) = v''$.

Claim 9: Every $g \in G$ is α_v for unique reduced $v \in W$ given by $v = g(\epsilon)$. *Proof*: Claims 6, 8.

In summary, we set up a bijection between W and G, $v \to \alpha_v$ in the forward direction and $g(\epsilon) \leftarrow g$ in the backward direction.

Claim 10: G is isomorphic to the group whose underlying set is W and whose group operation is "concatenate and reduce." *Proof*: Claims 1-9.

Claim 11: Let H be any group and let $f : X \to H$ be any function. Then, there is a unique HM $\phi : G \to H$ such that $\phi(\alpha_x) = f(x)$ for $x \in X$. *Proof*: If $v = x_1^{t_1} \dots x_t^{n_t}$ is reduced, then we must have must have $\phi(\alpha_v) = \phi(\alpha_{x_1}^{n_1} \dots \alpha_{x_t}^{n_t}) = f(x_1)^{n_1} \dots f(x_t)^{n_t}$ by iteration on the HM property. Indeed, the function given by that definition is a HM from the HM and reduction axioms. So there is always a unique HM.