21-237: Math Studies Algebra I October 7, 2022

Lecture 17 : Universal properties

Lecturer: James Cummings

Scribe: Rajeev Godse

1 Philosophy

For a set X, recall that we built G with $(g_x : x \in X)$, $g_x \in G$. For all groups H and all functions $f : X \to H$, there is a unique HM $\phi : G \to H$ such that $\phi(g_x) = f(x)$ for all $x \in X$.

This property is what we call in mathematics a **universal property**.

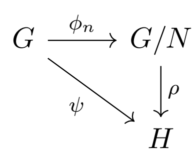
Uniqueness for G: Suppose G', $(g'_x : x \in X)$, $g'_x \in G'$ with the same "universal property," namely, for all groups H and all functions $f : X \to H$, there is a unique HM $\phi' : G' \to H$ such that $\phi(g_x) = f(x)$ for all $x \in X$. Then, there is exactly one isomorphism $\psi : G \simeq G'$ such that $\psi(g_x) = g'_x$ for all $x \in X$.

Proof: By the universal property of G, there is unique HM $\psi : G \to G'$ such that $\psi(g_x) = g'_x$ for all $x \in X$. By the universal property of G', there is a unique $\psi' : G' \to G$ such that $\psi'(g'_x) = g_x$ for all $x \in X$. $\psi' \circ \psi$ is a HM from G to G that fixes g_x for all $x \in X$. id_G also does this, so by the uniqueness in the universal property for G, $\psi' \circ \psi = \mathrm{id}_G$. Similarly, $\psi \circ \psi' = \mathrm{id}_{G'}$.

Example: Let $X = \{a, b\}$. Let G be the free group on X with $g_a, g_b \in G$ corresponding to a, b. Let H be any group such that $H = \langle h_0, h_1 \rangle$. Let $f : X \to H$ where $f(a) = h_0$, $f(b) = h_1$. By the universal property, there is unique $\phi : G \to H$ such that $\phi(g_a) = h_0$, $\phi(g_b) = h_1$. Clearly, ϕ is surjective, i.e. $\mathrm{im}(\phi) = H$. By the first isomorphism theorem, then $\frac{G}{\ker(\phi)} \simeq H$.

2 A nice property

General fact: Let G be any group, $N \triangleleft G$, $\phi_N : G \to G/N$ the quotient HM. As we know, $N = \ker(\phi_N)$. Let $\psi : G \to H$ be any HM such that $N \leq \ker(\psi)$. Then, there is a unique HM $\rho : G/N \to H$ such that $\psi = \rho \circ \phi_N$. The statement above corresponds to the below commutative diagram:



Intuition: ϕ_N is the "most general" HM whose kernel contains N.

Proof: For such a ρ, we must have that $\rho(gN) = \psi(g)$ for commutativity to hold.

Consider $\rho = gN \mapsto \psi(g)$. It's well-defined since elements that share the same coset of N also share a coset of $\ker(\psi) \geq N$. Then, the first isomorphism theorem gives us that they are mapped to the same value in ψ. The universal property clearly holds.

3 Equations

Let G be a group. An **equation** in G is an equality $h_1^{n_1} \dots h_t^{n_t} = 1$ for $h_i \in G$, $n_i \in \mathbb{Z}$.

Let G be free on X with generating elements $(g_x \in G : x \in X)$. Let E be a set of reduced words. For each $e \in E$, say $e = x_1^{n_1} \dots x_t^{n_t}$, let $g_e = g_{x_1}^{n_1} \dots g_{x_t}^{n_t} \in G$. Let N be the normal subgroup of G generated by $\{g_e : e \in E\}$.

Let $G_E = G/N$, $\phi_E : G \to G_E$ the quotient map. For each $x \in X$, let $\overline{g_x} \in G_E$ with $\overline{g_x} = g_x N$.

Remarks:

1. Since G is generated by $\{g_x : x \in X\}$, it's easy to see G_E is generated by $\{\overline{g_x} : x \in X\}$.
2. For all $e \in E$, say $e = x_1^{n_1} \dots x_t^{n_t}$, the equation $\overline{g_{x_1}}^{n_1} \dots \overline{g_{x_t}}^{n_t} = 1$ holds in G_E.

Universal property for G_E: Let H be a group, let $f : X \to H$ be a function such that for all $e \in E$, say $e = x_1^{n_1} \dots x_t^{n_t}$, we have that $f(x_1)^{n_1} \dots f(x_t)^{n_t} = 1$ in H. Then, there is a unique HM $\rho : G_E \to H$, $\rho(\overline{g_x}) = f(x)$ for all $x \in X$.