21-237: Math Studies Algebra I October 10, 2022

Lecture 18 : Generators and relations

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1 Free group conventions

Recall that for set X we defined the set of reduced words W over X. Then, the free group on X was defined as the group of permutations of the set of reduced words, generated by permutations α_x for $x \in X$. Recall that α_x corresponds to prepending a word with x and reducing.

We also noted that the free group on X was isomorphic to the group whose underlying set is W, with group operation "concatenate & reduce," where α_x corresponds to the reduced word x^1.

Convention 1: We view the free group on X as consisting of words instead of permutations.

Convention 2 (dangerous): We identify each $x \in X$ with the corresponding word x^1.

With conventions 1 and 2, X is literally a subset of the free group $\mathrm{Fr}(X)$.

Freeness now says that for any group H and any function $f : X \to H$, there is a unique HM $\phi : Fr(X) \to H$, $\phi \upharpoonright X = f$.

2 Generators and relations

Last time, we had a set E of reduced words, which coded the equations we wanted to hold in our free group. We also had N, the least normal subgroup containing E.

We also defined $G = \mathrm{Fr}(X)$, $G_E = G/N$, and $\phi_E : G \to G_e$ the quotient HM, i.e. $\phi(w) = wN$.

Let $\overline{x} = \phi(x) = xN$ for each $x \in X$. It's easy to see the following:

1. G_E is generated by $\{\overline{x} : x \in X\}$.

   Proof: gN is the product of the cosets of the elements of X that multiply to g in G. □

2. If $w = x_1^{n_1} \dots x_t^{n_t} \in E$, then $\overline{x}_1^{n_1} \dots \overline{x}_t^{n_t}$.

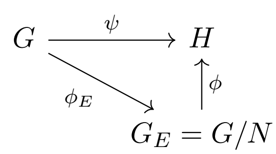
   Proof: $w \in N$ and $\ker(\phi_E) = N$ so $\phi_E(w) = \overline{x}_1^{n_1} \dots \overline{x}_t^{n_t} = 1$. □

3. For all groups H and all $f : X \to H$ such that for all words $x_1^{n_1} \dots x_t^{n_t} \in E$, $f(x_1)^{n_1} \dots f(x_t)^{n_t} = 1$, there is a unique HM $\phi : G_E \to H$ such that $\phi(\overline{x}) = f(x)$ for all $x \in X$.

   Proof: There is at least one ϕ: $\phi(\overline{x}_1^{t_1} \dots \overline{x}_t^{n_t}) = f(x_1)^{t_1} \dots f(x_t)^{n_t}$.

   By "freeness" of G, there is a unique HM $\psi : G \to H$ such that $\psi \upharpoonright X = f$: $\psi(x_1^{n_1} \dots x_t^{n_t}) = f(x_1)^{n_1} \dots f(x_t)^{n_t}$. By hypothesis on f, $E \subseteq \ker(\psi)$. As $\ker(\psi) \triangleleft G$ and N is the least normal subgroup containing E, $N \leq \ker(\psi)$.

   By a theorem from last lecture, there is a unique HM $\phi : G_E \to H$ such that $\psi = \phi \circ \phi_E$:



   In particular, for $x \in X$, $f(x) = \psi(x) = \phi(\phi_E(x)) = \phi(xN) = \phi(\overline{x})$. □

2.1 Examples

Let $n > 1$. Let $X = \{a, b\}$, $E = \{a^2, b^n, abab\}$.

G_E in this case is $\text{Fr}(\{a, b\})/N$, G_e is generated by $\overline{a}$, $\overline{b}$, where in G_E, $\overline{a}^2 = 1$, $\overline{b}^n = 1$, $\overline{b}^{\overline{a}} = \overline{b}^{-1}$.

Claim 1: Every element of G_E can be written as $\overline{a}^i\overline{b}^j$ for $0 \leq i < 2$, $0 \leq j < n$.

Proof: We can move each $\overline{a}$ to the left since $\overline{b}\overline{a} = \overline{a}\overline{a}\overline{b}\overline{a} = \overline{a}\overline{b}^{n-1}$. Then we can take powers mod 2 and n owing to the order of the elements. □

Thus, $|G_E| \leq 2n$, over choices of i and j.

Let H be the dihedral group of symmetries of a regular n-gon.

Let $c \in H$ be rotation by $\frac{2\pi}{n}$. Let $d \in H$ be some reflection.

Observe: $d^2 = 1$, $c^n = 1$, and $dcd^{-1} = c^{-1}$. There is a unique $\phi : G_E \to H$ such that $\phi(\overline{a}) = d$, $\phi(\overline{b}) = c$. As $H = \langle c, d \rangle$, ϕ is surjective. $|G_E| \leq 2n$ and $|H| = 2n$, so ϕ is an IM.