21-237: Math Studies Algebra I October 12, 2022

Lecture 19 : Cayley graphs

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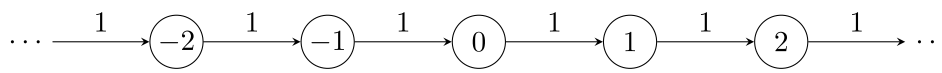
1 (Right) Cayley graphs

Given a group G and subsets $S \subseteq G$, the corresponding **(right) Cayley graph**, $\text{Cay}(G, S)$, is the labeled, directed graph on vertex set G with an edge labeled by s from g to h when $h = gs$ for $s \in S$.

Examples:

1. $G = \mathbb{Z}$, $S = \{1\}$



2. $G = \mathbb{Z}$, $S = \{2, 3\}$. Similar to above.
3. $G = \langle a, b : a^2 = 1, b^5 = 1, aba = b^{-1} \rangle, S = \{a, b\}$. You can work it out for yourself, but we get two "circles" acting in opposite directions, connected at each vertex bidirectionally by reflections.
4. $G = \mathbb{F}_2 = \langle a, b \rangle, S = \{a, b\}$. The Cayley graph is an infinite 4-regular tree, as each reduced word represents a sequence of operations such that the inverse of a previous operation does not occur after that operation.
5. $G = \langle a, b : aba^{-1}b^{-1} = e \rangle, S = \{a, b\}$. The Cayley graph gives us good intuition that $G \simeq \mathbb{Z}^2$.

2 Commutators (again!)

Recall, for $g, h \in G$, their **commutator** is $[g, h] = ghg^{-1}h^{-1}$, and $[g, h] = 1 \iff gh = hg$ by extreme obviousness.

Further recall the **commutator subgroup**, or **derived subgroup**, $G' = [G, G]$, is $\langle [a, b] : a, b \in G \rangle$.

More generally, for $A, B \subseteq G$, define $[A, B] = \langle [a, b] : a \in A, b \in B \rangle$.

Lemma: Let G be a group, $H \leq G$. The following are equivalent:

(I) $H \triangleleft G$

(II) $[H, G] \leq H$

Proof: Suppose $H \triangleleft G$. Let $h \in H$, $g \in G$. $[h, g] = h(h^{-1})^g \in H$. So $[H, G] \leq H$.

Now suppose $[H, G] \leq H$. Let $h \in H$, $g \in G$. $h^g = h[h^{-1}, g] \in H$. So $H \triangleleft G$. □

Certainly, we have $[G', G] \leq [G, G] = G'$. So by the lemma, $G' \triangleleft G$.

Claim: For any group G, G/G' is abelian.

Proof: Let $aG', bG' \in G/G'$. $(aG')(bG') = abG' = [a, b]baG' = ([a, b]G')baG' = bG'aG'$. □

Theorem: Let G be a group, $H \leq G$. The following are equivalent:

(I) $G' \leq H$

(II) $H \triangleleft G$ and G/H is abelian.

So G' is the least subgroup with abelian quotient.

Proof: Suppose $G' \leq H$. Certainly, $[H, G] \leq [G, G] = G' \leq H$. So from the lemma, $H \triangleleft G$.

Then, as above, $ab = [a, b]ba$, so $(aH)(bH) = abH = baH = (bH)(aH)$.

Now, suppose G/H is abelian. Then $\forall a, b \in G$, $abH = baH$. Equivalently, $aba^{-1}b^{-1}H = H$, i.e. $aba^{-1}b^{-1} \in H$. Hence, $[G, G] \in H$. $\square$

2.1 As a universal property

The **abelianization** of a group G is the quotient map $q : G \to G/G'$. It is characterized by the following *universal property*:

Given any abelian group A and any homomorphism $\phi : G \to A$, there exists a unique homomorphism $\psi : G/G' \to A$ such that $\phi = \psi \circ q$.

Proof: Note that by the first IM theorem, $G/\ker(\phi) \simeq \text{im}(\phi) \leq A$, so $G/\ker(\phi)$ is abelian. Accordingly, $G' \leq \ker(\phi)$. Thus, whenever $gG' = hG'$, we know that $\phi(g) = \phi(h)$ since $g^{-1}h \in G' \leq \ker(\phi)$. We may thus unambiguously define $\psi : G/G' \to A$ by $gG' \mapsto \phi(g)$. Certainly, it works, so ψ is unique. $\square$