21-237: Math Studies Algebra I August 31, 2022

Lecture 2 : More on Groups

Lecturer: James Cummings

Scribe: Rajeev Godse

1 Meta-note on structure

Algebra is a structuralist domain: we define various structures axiomatically. We discuss transformations and comparisons for the group structure.

2 Group isomorphisms

2.1 Definitions

Let $(G, *_1)$, $(G_2, *_2)$ be groups. An **isomorphism** (IM) from $(G_1, *_1)$ to $(G_2, *_2)$ is a bijection $\alpha : G_1 \simeq G_2$ such that $\alpha(g *_1 h) = \alpha(g) *_2 \alpha(h)$ for all $g, h \in G_1$.

We say groups G_1, G_2 are **isomorphic** if there exists an isomorphism from G_1 to G_2.

2.2 Examples

For $G_1 = (\mathbb{R}, +)$, $G_2 = (\{a : a \in \mathbb{R}, a > 0\}, \times)$, exp is an IM: $\forall x, y \in \mathbb{R}. \exp(x)\exp(y) = \exp(x + y)$.

2.3 Theorems

(a) For all G, G is isomorphic to G (via the identity).

(b) If G_1 is isomorphic to G_2 (via α), then G_2 is isomorphic to G_1 (via α^{-1}).

Short proof: Let $x, y \in G_2$. $\alpha^{-1}(x)\alpha^{-1}(y) = \alpha^{-1}(\alpha(\alpha^{-1}(x)\alpha^{-1}(y))) = \alpha^{-1}(xy)$.

(c) If G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3, then G_1 is isomorphic to G_3 (via composition of the isomorphisms).

Due to (a), (b), and (c), group isomorphism is a **equivalence relation**.

3 Subgroups

3.1 Definition

Let $G = (S, *)$ be a group. A **subgroup** of G is a subset $H \subseteq S$ such that (a) H is closed under the group operation of G, i.e. $\forall a, b \in H. ab \in H$; (b) $1 \in H$; (c) H is closed under inverse, i.e. $\forall a \in H. a^{-1} \in H$.

Key: H itself forms a group under $*$ restricted to $H \times H$. This can be easily verified.

$H = \{1\}$ is always a subgroup, and is called the **trivial subgroup**.

4 Group homomorphisms

4.1 Definition

Let G_1, G_2 be groups. A **homomorphism** (HM) from G_1 to G_2 is $\alpha : G_1 \to G_2$ such that $\alpha(gh) = \alpha(g)\alpha(h)$ for all $g, h \in G_1$.

4.2 Theorems

Let $\alpha : G_1 \to G_2$ be HM.

Fact 1: $\alpha(1) = 1$.

Lemma: In a group G, if $b = bb$, then $b = 1$. To see this, multiply by b^{-1} on both sides.

Proof: See that $\alpha(1)\alpha(1) = \alpha(11) = \alpha(1)$. By the lemma, $\alpha(1) = 1$.

Fact 2: For all $g \in G_1$, $\alpha(g^{-1}) = \alpha(g)^{-1}$.

Proof: $\alpha(g)\alpha(g^{-1}) = \alpha(gg^{-1}) = \alpha(1) = 1$. Inverses are unique, so $\alpha(g)^{-1} = \alpha(g^{-1})$

5 More examples of groups

If G is a group, then $\mathbf{Aut}(\mathbf{G})$ is a group under composition.

Let G_1, G_2 be groups. The **product group** $G_1 \times G_2$ is $\{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ with operation $(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2')$.

More generally, let $(G_i)_{i \in I}$ be an indexed family of groups. We can define $\prod_{i \in I} G_i$ in the obvious way: elements are $(g_i)_{i \in I}$, $g_i \in G_i$, with "coordinate-wise" operation and identity.