21-237: Math Studies Algebra I October 14, 2022

Lecture 20 : Categories

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1 Category theory

1.1 Definitions

To specify a **category** $\mathbb{C}$, we need:

(a) A collection of **objects** $\mathrm{Obj}(\mathbb{C})$

(b) A collection of **morphisms** (**arrows**) $\mathrm{Mor}(\mathbb{C})$

(c) For each object c in $\mathrm{Obj}(\mathbb{C})$, a morphism id_C

(d) For each arrow f in $\mathrm{Mor}(\mathbb{C})$, objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$

(e) For each pair of morphisms (f, g) such that $\mathrm{dom}(f) = \mathrm{cod}(g)$, a morphism $f \circ g$ with domain $\mathrm{dom}(g)$ and codomain $\mathrm{cod}(f)$

A category $\mathbb{C}$ must obey the following axioms:

(1) For all $c \in \mathrm{Obj}(c)$, $\mathrm{dom}(\mathrm{id}_c) = \mathrm{cod}(\mathrm{id}_c) = c$.

(2) For all $f : a \to b$, $f \circ \mathrm{id}_a = \mathrm{id}_b \circ f = f$.

(3) For all $f : c \to d$, $g : b \to c$, $h : a \to b$, $f \circ (g \circ h) = (f \circ g) \circ h$.

1.2 Examples

- Category of sets: the objects are sets, the arrows are functions.
- Category of groups: the objects are groups, the arrows are homomorphisms.
- Metric spaces: the objects are metric spaces, the arrows are isometries.
- For any poset[1] $(P, \leq)$, we can define a category $\mathbb{C}$ where $\mathrm{Obj}(\mathbb{C}) = P$, where if $p \leq q$, there is exactly one arrow from $p \to q$, and if $p \not\leq q$, there is no arrow from $p \to q$.
- Let $\mathbb{C}$ be any category. $\mathbb{C}^{op}$ (opposite category) is $\mathbb{C}$ with arrows reversed.
- (Small) categories: the objects are (small) categories and the arrows are functors[2].
- For categories $\mathbb{C}$, $\mathbb{D}$, the functor category $\mathbb{D}^{\mathbb{C}}$ has functors from $\mathbb{C}$ to $\mathbb{D}$ as objects and natural transformations[3] as arrows.

[1] A **poset** is a set P equipped with a binary relation $\leq$ that is reflexive ($p \leq p$), transitive ($p \leq q \leq r \implies p \leq r$), and antisymmetric ($p \leq q \wedge q \leq p \implies q = p$) for all $p, q, r \in P$. For any set X, $(\mathcal{P}(X), \subseteq)$ is a poset.

[2] Let $\mathbb{C}, \mathbb{D}$ be categories. A **functor** $F : \mathbb{C} \to \mathbb{D}$ sends each $c \in \mathrm{Obj}(\mathbb{C})$ to $F(c) \in \mathrm{Obj}(\mathbb{D})$ and each $f : a \to b$ in $\mathrm{Mor}(\mathbb{C})$ to $F(f) : F(a) \to F(b)$ in $\mathrm{Mor}(\mathbb{D})$ such that $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ and $F(f \circ g) = F(f) \circ F(g)$.

[3] For functors $F, G : \mathbb{C} \to \mathbb{D}$, a **natural transformation** $\nu : F \to G$ is a family $(\nu_a : F(a) \to G(a))_{a \in \mathrm{Obj}(\mathbb{C})}$ such that for all $(f : a \to a') \in \mathrm{Mor}(\mathbb{C})$, $\nu_{a'} \circ F(f) = G(f) \circ \nu_a$.