21-237: Math Studies Algebra I October 24, 2022

Lecture 21 : Zorn's lemma and introduction to rings

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1 Posets and properties

Recall that a **poset** $(\mathbb{P}, \leq)$ has that for all $a, b, c \in \mathbb{P}$

1. $a \leq a$
2. $a \leq b \wedge b \leq a \implies a = b$
3. $a \leq b \wedge b \leq c \implies a \leq c$

For any set X, $(\mathcal{P}(X), \subseteq)$ is a poset.

For poset $(\mathbb{P}, \leq)$ and $p, q \in \mathbb{P}$, we say $p < q$ if $p \leq q$ and $p \neq q$.

A poset $(\mathbb{L}, \leq)$ is a **loset** (linear ordering, total ordering) if for all $a, b \in \mathbb{L}$, $a \leq b$ or $b \leq a$ [equivalently, $a < b$, $a = b$, or $b < a$].

For poset $(\mathbb{P}, \leq)$, a set $C \subseteq \mathbb{P}$ is a **chain** if C is linearly ordered by the restriction of $\leq$ to C.

For poset $(\mathbb{P}, \leq)$, we say $p \in \mathbb{P}$ is **maximal** if there is no $q \in \mathbb{P}$ such that $p < q$.

For poset $(\mathbb{P}, \leq)$, we say $p \in \mathbb{P}$ is **maximum** if $q \leq p$ for all $p \in \mathbb{P}$.

Facts:

1. If p is maximum, p is maximal.
2. There is at most one maximum element.

For poset $(\mathbb{P}, \leq)$, $X \subseteq P$, we say X is **bounded** if there is $p \in \mathbb{P}$ such that $x \leq p$ for all $x \in X$.

2 Zorn's Lemma

Lemma: Let $(\mathbb{P}, \leq)$ be a poset in which every chain is bounded. Then for every $p \in \mathbb{P}$, there is $q \in \mathbb{P}$ such that $p \leq q$ and q is maximal.

"Proof": Let $p_0 = p$. If p_0 maximal, then we are done. Otherwise, there is p_1, where $p_0 < p_1$. If p_1 is maximal, we are done. Otherwise, find p_2, where $p_1 < p_2$. Suppose that continuing in this fashion, p_n exists for all $n \in \mathbb{N}$. Clearly, $\{p_n : n \in \mathbb{N}\}$. Let p_ω be such that $p_n < p_\omega$ for all $n \in \mathbb{N}$, existing since every chain is bounded. Continue: $p_\omega, p_{\omega+1}, p_{\omega+2}, \ldots$. If $p_{\omega+n}$ exists for all $n \in \mathbb{N}$, choose $p_{\omega+n} < p_{\omega+\omega}$ for all $n \in \mathbb{N}$. Because $\mathbb{P}$ is a set, this process must terminate. □

(Cautionary) example: Let $\mathbb{P}$ be the set of countable subsets of $\mathbb{R}$, ordered by $\subseteq$. Any countable chain in $\mathbb{P}$ is bounded (by its union). However, $\mathbb{P}$ has no maximal element. Indeed, we could not have applied Zorn's lemma since uncountable chains of this poset are not bounded in general.

3 Rings

A ring is a set R equipped with 2 binary operations: $+$ and $\times$. The following axioms must hold:

1. $(R, +)$ is an abelian group. 0 is the identity of $(R, +)$, and $-r$ is the inverse of r.
2. $\times$ is associative.
3. $\times$ distributes over $+$ from either side. That is, $a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$. Equivalently, left and right application of $\times$ with a fixed element is a group homomorphism of $(R, +)$.