21-237: Math Studies Algebra I October 31, 2022

Lecture 23 : Modules

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1 Unital rings

Rings with 1 (unital rings) are rings with a 2-sided identity for multiplication.

By convention, in the category of unital rings, a morphisms is a ring HM $\phi : R \to S$ with the additional property that $\phi(1_R) = 1_S$.

We also change the "meaning" of subring, so R only counts as a subring of S if $1_R = 1_S$, which ensures that they are subobjects in the category of unital rings.

Other terminology about ideals will be unchanged.

Note: If I is a left (or right) ideal of a unital ring R, $1 \in I \iff I = R$.

2 Modules

Let R be a unital ring. A left R-module is M equipped with 2 binary operations: $+$ on M and a map "scalar multiplication" from $R \times M$ to M such that

(1) $(M, +)$ is an abelian group.

(2) For $r_1, r_2 \in R$, $m \in M$, $(r_1 + r_2)m = r_1 m + r_2 m$. For $r \in R$, $m_1, m_2 \in M$, $r(m_1 + m_2) = rm_1 + rm_2$.

(3) For $r_1, r_2 \in R$, $m \in M$, $r_1(r_2 m) = (r_1 r_2)m$.

(4) For $m \in M$, $1m = m$.

(5) For $m \in M$, $0m = m$.

Examples:

(1) $R = \mathbb{R}$, $M = \mathbb{R}^n$.

(2) R unital ring, $M = (R, +)$.

(3) R unital ring, $(I, +)$ for some left ideal I of R.

(4) Where M, N are left R-modules for some unital R, $M \oplus N = \{(m, n) : m \in M, n \in N\}$ with the natural component-wise operations, called the **direct sum** of M and N, forms a left R-module.

To define right R-modules, just take the above definitions and move the "scalar multiplication" operation to take elements of R on the right, everything works symmetrically.

For unital ring R, a **2-sided** R-module is M equipped with $+$ such that

(1) $(M, +)$ is a left R-module

(2) $(M, +)$ is a right R-module

(3) $r(ms) = (rm)s$ for $r, s \in R$, $m \in M$.

Example: If I is a 2-sided ideal of a unital ring R, I is a 2-sided R-ideal.

3 Ordering ideals

Now, fix a unital ring R and look at $\{I : I \text{ left ideal of } R\}$ as a poset under inclusion. This has a minimum element of 0 and a maximum element of R.

Let $a \in R$. The least left ideal containing a is $Ra = \{ra : r \in R\}$. *Note*: As R is unital, $a = 1a \in Ra$.