21-237: Math Studies Algebra I November 7, 2022

Lecture 26 : Field of fractions

Lecturer: James Cummings

Scribe: Rajeev Godse

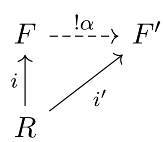
1 Subrings

If R is a subring of S then $j : R \to S$ is the inclusion map $[j(r) = r]$ is an injective ring homomorphism. Conversely, if R, S are rings and we have $i : R \to S$ an injective ring homomorphism, we have that $i[R]$ is a subring of S and $R \simeq i[R]$.

In most algebraic settings, a **monomorphism** (MM) is an injective homomorphism, and a **epimorphism** (EM) is a surjective homomorphism.

2 Fieldifying IDs

Theorem: Let R be an ID. Then there exists field F and MM $i : R \to F$ such that for all fields F' and all MM's $i' : R \to F'$, there is a unique HM $\alpha : F \to F'$ such that $\alpha \circ i = i'$.



In general, a ring HM $\phi : A \to B$ is MM $\iff$ $\ker(\phi) = 0$. If A, B fields, $\ker(\phi)$ is an ideal of A, $\phi(1_A) = 1_B \neq 0_B$, so $\ker(\phi) \neq A$ and thus is 0 and ϕ MM. So every ring HM from a field is MM.

Thus, α is MM. Philosophically, F is the least field containing a copy of R, since any field containing a copy of R also contains a copy of F.

Proof: Let $X = \{(a, b) : a \in R, b \in R \setminus \{0\}\}$.

Define the binary relation E such that $(a_0, b_0)E(a_1, b_1) \iff a_0b_1 = a_1b_0$.

Reflexivity and symmetry are easy. For transitivity, suppose $(a_0, b_0)E(a_1, b_1)E(a_2, b_2)$, i.e. $a_0b_1 = a_1b_0$ and $a_1b_2 = a_2b_1$. Then, $(a_0b_2)b_1 = (a_0b_1)b_2 = (a_1b_0)b_2(a_2b_0)$. Then, $(a_0b_2 - a_2b_0)b_1 = 0$, $b_1 \neq 0$, and R is ID, so $a_0b_2 - a_2b_0 = 0$, i.e. $a_0b_2 = a_2b_0$.

For $(a, b) \in X$ define a/b as the E-equivalence class of (a, b).

Let $F = \{a/b : (a, b) \in X\}$. We attempt to define $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. It's an easy exercise to verify that these definitions make F into a commutative unital ring with $1_F = \frac{1}{1}$ and $0_F = \frac{0}{1}$. Slightly subtly, $\frac{a}{b} = \frac{0}{1} \iff a = 0$. As a consequence, we can define $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$ for $\frac{a}{b} \neq 0_F$ since $a \neq 0$. This also gives us that $0_F \neq 1_F$. Thus, F is a field.

Define $i : r \in R \mapsto \frac{r}{1} \in F$. i is easily a ring HM. $\ker(i) = \{r \in R : r/1 = 0\} = \{0\}$, so R is injective

Note: for $(a, b) \in X$, $\frac{a}{b} = i(a)i(b)^{-1}$ since i is MM.

Let $i' : R \to F'$ be an MM for some field F'.

If α exists, for $(a, b) \in X$, $\alpha(a/b) = \alpha(i(a))\alpha(i(b))^{-1} = i'(a)i'(b)^{-1}$.

NB: α is injective, so $\alpha(a/b) \neq 0$ unless $a = 0$. Similar for i'.

We can also verify that this definition of α is well-defined and forms an MM. So α must be unique. □