21-237: Math Studies Algebra I November 11, 2022

Lecture 28 : Noetherian modules

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1 Modular definitions

For N a module, we say that $M \leq N$ (M is a submodule of N) if $M \leq (N, +)$ and M is closed under scalar multiplication. The submodules of R acted on by itself are just its ideals.

If $M \leq N$, we can form the quotient module $\frac{N}{M}$ and quotient HM $\phi_M : n \in N \to n + M$ and $r(n + M) = rn + M$.

$\phi : M \to N$ is an R-module HM if $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$ and $\phi(rm) = r\phi(m)$ (ϕ is R-linear). Equivalently, $\phi\left(\sum_{i=1}^{t} r_i m_i\right) = \sum_{i=1}^{t} r_i \phi(m_i)$.

If $\phi : M \to N$ is HM, $\ker(\phi) = \{m \in M : \phi(m) = 0\} \leq M$, $\operatorname{im}(\phi) = \phi[M] = \{n : \exists m \in M, n = \phi(m)\}$.

Then, first IM theorem holds: $\frac{M}{\ker(\phi)} \simeq \phi[M]$ via $m + \ker(\phi) \leftrightarrow \phi(m)$.

If N is a module and $X \subseteq N$, the least submodule containing X is the span of X, which is $\{\sum_{i=1}^{t} r_i x_i : r_i \in R, x_i \in X, t \in \mathbb{N}\}$.

2 Noetherian modules

Recall: We defined module M to be **Noetherian** if all submodules are finitely generated.

Let M be R-module. The following are equivalent:

(1) M is Noetherian.

(2) For any sequence $(M_n)_{n\in\mathbb{N}}$ of submodules of M such that $M_0 \subseteq M_1 \subseteq \ldots$, there is m such that $M_m = M_n$ for $n \geq m$. This is known as the **ascending chain condition**, abbreviated ACC.

(3) For any non-empty set X of submodules of M, there is $N \in X$ maximal under inclusion, i.e. for all $\overline{N} \in X$, if $N \subseteq \overline{N}$, then $N = \overline{N}$.

Proof: (1) $\implies$ (2): Let $M_0 \subseteq M_1 \subseteq \ldots$, each submodule $M_i \leq M$.

Let $\overline{M} = \bigcup_{i\in\mathbb{N}} M_i$. It's easy to see that $\overline{M} \leq M$, so $\overline{M}$ is finitely generated, i.e. there exist $b_1, \ldots, b_t \in \overline{M}$ such that $\overline{M}$ is the span of b_j's. So there is m such that $b_j \in M_m$ for all $1 \leq j \leq t$.

$\overline{M}$ = span of b_j's $\subseteq M_m \subseteq M_n \subseteq \overline{M}$ so $M_n = M_m$ for all $n \geq m$. □

(2) $\implies$ (1): Let $N \leq M$, N not finitely generated. Choose $m_0 \in N$, $m_0 \neq 0$, let $M_0 = Rm_0$. $M_0 \neq N$ as M_0 is finitely generated. For $i \geq 0$ assume M_i is finitely generated and thus not equal to N. Let $m_{i+1} \in N \setminus M_i$ and let $M_{i+1} = M_i + rm_{i+1}$. M_{i+1} is also not finitely generated. So we have an infinite chain that keeps getting bigger (we keep adding new elements), so it is not eventually constant. □

(2) $\implies$ (3): Let X be a non-empty set of submodules with no maximal element. Let $M_0 \in X$. As M_0 not maximal, find $M_1 \in X$, $M_0 \subset M_1$, also not maximal. In general, for non-maximal $M_i \in X$, $M_i \subset M_{i+1}$. So the chain is not eventually contant. □

(3) $\implies$ (2): Let $(M_i)_{i\in\mathbb{N}}$, $M_i \subseteq M_{i+1}$, not eventually constant. Let $X = \{M_i : i \in \mathbb{N}\}$. The existence of a maximal element would show that the sequence is eventually constant. □

3 Non-examples

Recall: A ring R is Noetherian if R is Noetherian as an R-module.

Example: Let's look for a **non-Noetherian** ring. Let $R = \mathbb{Z}[x_0, x_1, x_2, \ldots]$, the ring of polynomials in infinite set of variables with integer coefficients. Note that each polynomial is still finite, so we can write $R = \mathbb{Z} \cup \mathbb{Z}[x_0] \cup \mathbb{Z}[x_0, x_1] \cup \mathbb{Z}[x_0, x_1, x_2] \cup \ldots$

Let I be the ideal spanned by $\{x_i : i \in \mathbb{N}\}$ = ideal of $f \in R$ with constant term zero.

Claim: I not finitely generated. *Proof*: If not, let $f_1, \ldots f_t \in I$ be such that I is the span of $f_1, \ldots, f_t$. Let x_k be variable not appearing in any f_j. As $x_k \in I$, there exist $g_1, \ldots, g_t \in R$, $x_k = \sum_{i=1}^{t} g_i f_i$. If we let $x_k = 1$ and all other $x_n = 0$, all f_i's are 0, we get $0 = 1$. □