**21-237: Math Studies Algebra I** November 28, 2022

Lecture 33 : Uniquely factorizing polynomials (continued)

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# 1 Finishing the Proof

For UFD $R$ with field of fractions $K$, *recall*:

(1) We say $f \in R[x]$, $f \neq 0$ is **primitive** if a gcd of the coefficients of $f$ is a unit, or equivalently, there is no prime $p$ in $R$ such that $p \mid f$ in $R[x]$.

(2) The units of $R$ are units in $R[x]$. $p \in R$ is irreducible in $R \iff p$ is irreducible in $R[x]$.

(3) **Gauss's Lemma**: If $f, g \in R[x]$ primitive, then $fg$ is primitive.

(4) The units of $K[x]$ are $K \setminus \{0\}$. If $f \in K[x]$ non-zero, then $f$ has an associate in $K[x]$ that is primitive in $R[x]$.

**Strategy**: $R \subseteq R[x] \subseteq K[x]$ and $R, K[x]$ are UFDs.

**Lemma**: Let $f \in R[x]$ be primitive, $\deg(f) > 0$. $f$ is irreducible in $R[x] \iff f$ irreducible in $K[x]$.

*Proof*: Suppose $f$ is irreducible in $R[x]$. Let $g, h \in K[x]$ and suppose $f = gh$.

We can find $C_0, D_0, C_1, D_1 \in R$, $D_0, D_1 \neq 0$ such that $(C_0/D_0)g$ and $(C_1/D_1)h$ are primitive elements of $R[x]$. Then, $(C_0/D_0)(C_1/D_1)f = ((C_0/D_0)g)((C_1/D_1)h)$. By Gauss's Lemma, this product of two primitives is primitive in $R[x]$.

We can also get the equation in $R[x]$ that $C_0C_1f = D_0D_1((C_0/D_0)g)((C_1/D_1)h)$.

As $f$ is primitive, $C_0C_1$ is the content of the LHS of the above equation. As $(((C_0/D_0)g)((C_1/D_1)h)$ is primitive, $D_0D_1$ is the content of the RHS. So $C_0C_1$, $D_0D_1$ are associates in $R$. Adjusting $C_0$ by a unit if necessary which leaves intact its desired properties, we may assume WLOG that $C_0C_1 = D_0D_1 \neq 0$.

Since $R[x]$ is ID, we have that $f = ((C_0/D_0)g)((C_1/D_1)h)$ in $R[x]$. Since $f$ is irreducible, one of its factors must be a unit in $R[x]$, i.e. a unit in $R$, so either $f$ or $g$ is a non-zero element in $K$, i.e. a unit in $K[x]$. Thus, $f$ is irreducible in $K[x]$.

Conversely, suppose $f$ is irreducible in $K[x]$. Let $f = gh$ in $R[x]$.

$g, h \in K[x]$, so WLOG, $g$ is a unit in $K[x]$, i.e. a unit in $K$.

$g \in R[x] \cap (K \setminus \{0\}) = R \setminus \{0\}$, so $g$ is non-zero in $R$.

$f = gh$, so $g$ divides all the coefficients in $f$. Since $f$ is primitive, $g$ must be a unit in $R$, i.e. a unit in $R[x]$. Thus, $f$ is irreducible in $R[x]$. $\square$

We now want to show that factorizations in $R[x]$ exist, and are unique.

**Existence**: Let $f \in R[x]$, $f$ is nonzero, nonunit.

Where $c$ is the content of $f$, $f = cf_0$ for primitive $f_0$.

If $c$ is a unit, that's great, we can fold it into the factorization of $f_0$. Otherwise, we can factor $c$ into irreducibles in $R$, which are also irreducibles in $R[x]$.

Now, let's view $f_0$ as a polynomial in $K[x]$. If $f_0$ is a unit in $K[x]$, then $f_0 \in R$, so great, factor it in $R$. Otherwise, factor $f_0$ into irreducibles $h_1, \ldots, h_t$ in $K[x]$.

For each $i$, let $H_i \in R[x]$ be a primitive associate of $h_i$ in $K[x]$. Then, $F_0 = \prod_i H_i$ is an associate of $f_0$ in $K[x]$. $F_0$, the product of primitives in $R[x]$ is primitive in $R[x]$. So WLOG, $F_0 = f_0$, since they are both primitive, and so the fraction relating them in $K[x]$ must be a unit in $R[x]$.

Since each $H_i$ is primitive and $\deg(H_i) > 0$, each $H_i$ is irreducible in $R[x]$. So indeed, we have a factorization of $f$.

**Uniqueness**: Let $f \in R[x]$ be nonzero nonunit. Fix 2 factorizations of $f$ into irreducibles in $R[x]$.

Irreducibles in $R[x]$ are either irreducible in $R$ or non-zero degree and primitive.

Let $f = C_1 \dots C_s g_1 \dots g_t = D_1 \dots D_u h_1 \dots h_v$, where $C_i, D_i$ are irreducible in $R$ and $g_i, h_i$ are irreducible in $R[x]$ with nonzero degree, meaning they are also primitive.

Since $\prod_i g_i$, $\prod_i h_i$ are primitive, WLOG (up to associates), $\prod_i C_i = \prod_i D_i$. As $R$ UFD, $s = u$ and $C_i$'s are equal to $D_i$'s up to permutation and associates in $R$ and thus in $R[x]$.

Then, as $R[x]$ is ID, $\prod_i g_i = \prod_i h_i$. As $g_i, h_i$ are primitive, deg $> 0$, and irreducible in $R[x]$, they are irreducible in $K[x]$. As $K[x]$ is a UFD, $t = v$ and $g$'s and $h$'s are equal up to associates (in $K[x]$) and permutation. Since they are all primitives, they are also associates in $R[x]$. $\square$